Hochschule RheinMain

Programming Language Foundations

03 Lambda Calculus

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Contents

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Introduction

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The untyped lambda calculus

- is a foundational model of computation
- is the core of functional programming languages

Lambda notation is used in other settings too:

- non-functional programming languages like Java or Python have introduced functional concepts and lambda expressions
- in mathematics, lambda notation is used to represent function
- we use it later for denotational semantics of an imperative programming language

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SYNTAX OF THE LAMBDA CALCULUS

- Syntax of expressions
- Free and bound variables
- Capture-avoiding substitution
- Contexts

Expressions

Expressions

$$\mathbf{Expr} ::= V \mid \lambda V.\mathbf{Expr} \mid (\mathbf{Expr}\ \mathbf{Expr})$$

where V is a non-terminal for variables

Explanations:

$\lambda x.s$	is an abstraction	= an anonymous function, λx binds x in body s
$(s\ t)$	is an application	= expression s is applied to argument t

Examples:

- $\lambda x.x$ is the identity function, like $id(x) = x$, but anonymous
- $((\lambda x.x)\ z)$ represents $id(z)$
- $((\lambda x.x)\ (\lambda x.x))$ represents $id(id)$
- $\lambda x.\lambda y.x$ represents $f(x, y) = x$

Conventions

To omit parentheses, we use the following conventions:

- application is left-associative:

  $s\ t\ r$ means $((s\ t)\ r)$ and not $(s\ (t\ r))$

- the body of an abstraction extends as far as possible:

  $(\lambda x.s\ t$ means $\lambda x.(s\ t)$ and not $((\lambda x.s)\ t))$

- abbreviation:

  We write $\lambda x_1, \ldots, x_n.t$ for the nested abstractions $\lambda x_1.(\lambda x_2.\ldots.(\lambda x_n.t)\ldots)$.

Examples

Some prominent expressions:

$I := \lambda x.x$	(identity)
$K := \lambda x.\lambda y.x$	(projection to first argument)
$K_2 := \lambda x.\lambda y.y$	(projection to second argument)
$\Omega := (\lambda x.(x\ x))\ (\lambda x.(x\ x))$	(diverging expression)
$Y := \lambda f.(\lambda x.(f\ (x\ x)))\ (\lambda x.(f\ (x\ x)))$	(call-by-name fixpoint combinator)
$Z := \lambda f.(\lambda x.(f\ \lambda z.(x\ x)\ z))\ (\lambda x.(f\ \lambda z.(x\ x)\ z))$	(call-by-value fixpoint combinator)
$S := \lambda x.\lambda y.\lambda z.(x\ z)\ (y\ z)$	(S-combinator)

Free and Bound Variables

$$\begin{aligned} FV(x) &= x & BV(x) &= \emptyset \\ FV(\lambda x.s) &= FV(s) \setminus \{x\} & BV(\lambda x.s) &= BV(s) \cup \{x\} \\ FV(s\ t) &= FV(s) \cup FV(t) & BV(s\ t) &= BV(s) \cup BV(t) \end{aligned}$$

Example: $s = (\lambda x.\lambda y.\lambda w.(x\ y\ z))\ x$

Free variables $FV(s) = \{x, z\}$ and $BV(s) = \{x, y, w\}$.

Closed and open expressions:

- t is closed (or a program) if $FV(t) = \emptyset$
- otherwise t is open

Occurrence x in t is bound if it is in scope of a binder λx, otherwise it called free

Example: $((\lambda x.\lambda y.\lambda w.(\underbrace{x}_{\text{bound}}\ \underbrace{y}_{\text{bound}}\ \underbrace{z}_{\text{free}}))\ \underbrace{x}_{\text{free}})$

Substitution

Definition (Capture-Avoiding Substitution)

If $BV(s) \cap FV(t) = \emptyset$, then $s[t/x]$ is s where all free occurrences of x are replaced by t:

$$
\begin{array}{lcl}
x[t/x] & = & t \\
y[t/x] & = & y, \text{ if } x \neq y \\
(\lambda y.s)[t/x] & = & \begin{cases} \lambda y.(s[t/x]) & \text{if } x \neq y \\ \lambda y.s & \text{if } x = y \end{cases} \\
(s_1\ s_2)[t/x] & = & (s_1[t/x]\ s_2[t/x]) \\
(s_1\ s_2)[t/x] & = & (s_1[t/x]\ s_2[t/x])
\end{array}
$$

Example: $(\lambda x.z\ x)[(\lambda y.y)/z] = (\lambda x.((\lambda y.y)\ x))$.

Without the side condition: $(\lambda x.z\ x)[\lambda y.x/z]$ would lead to $\lambda x.((\lambda y.x)\ x)$

Contexts

Contexts C: Expressions with one hole $[\cdot]$

$$\mathbf{Ctxt} ::= [\cdot] \mid \lambda V.\mathbf{Ctxt} \mid (\mathbf{Ctxt}\ \mathbf{Expr}) \mid (\mathbf{Expr}\ \mathbf{Ctxt})$$

$C[s]$ is an expression: it is C where the hole is replaced by s

This may capture variables, e.g. for context $C = \lambda x.[\cdot]$ and expression $\lambda y.x$:

$$C[\lambda y.x] = \lambda x.(\lambda y.x).$$

α-Renaming and β-Reduction

- Renaming of variables
- Distinct variable convention
- Substitution (with renaming)
- β-reduction, contextual closure

Consistent Renaming of Variables

A single α-renaming-step is of the form:

$$C[\lambda x.s] \xrightarrow{\alpha} C[\lambda y.s[y/x]] \ \text{ if } y \notin BV(C[\lambda x.s]) \cup FV(C[\lambda x.s])$$

The reflexive-transitive closure of $\xrightarrow{\alpha} \cup \xleftarrow{\alpha}$ is called α-equivalence and written $s =_\alpha t$.

We identify α-equivalent expressions (and write $s = t$ also if $s =_\alpha t$)

The Distinct Variable Convention

To avoid naming conflicts, we assume the following convention:

Distinct Variable Convention (DVC)

In any expression s, bound and free variables are disjoint, i.e. $BV(s) \cap FV(s) = \emptyset$, and all variables on binders are pairwise distinct.

The convention can be obeyed by using α-renamings.

Example

$(y\ (\lambda y.((\lambda x.(x\ \lambda x.x))\ (x\ y))))$ violates the DVC

since x and y occur free and bound and x occurs twice at a binder

apply α-renamings to satisfy the DVC:

$$\begin{array}{ll}
& (y\ (\lambda y.((\lambda x.(x\ \lambda x.x))\ (x\ y)))) \\
\xrightarrow{\alpha} & (y\ (\lambda y_1.((\lambda x.(x\ \lambda x.x))\ (x\ y_1)))) \\
\xrightarrow{\alpha} & (y\ (\lambda y_1.((\lambda x_1.(x_1\ \lambda x.x))\ (x\ y_1)))) \\
\xrightarrow{\alpha} & (y\ (\lambda y_1.((\lambda x_1.(x_1\ \lambda x_2.x_2))\ (x\ y_1))))
\end{array}$$

Substitution with Renaming

Substitution $s[t/x]$ without side condition:

Definition (Substitution)

If $BV(s) \cap FV(t) = \emptyset$, then $s[t/x]$ is the capture-avoiding substitution.

Otherwise, let $s' =_\alpha s$ such that s' fulfills the DVC.

Then $BV(s') \cap FV(t) = \emptyset$ holds.

Then let $s[t/x] = s'[t/x]$ using capture-avoiding substitution for $s'[t/x]$.

Note: $s[t/x]$ may not satisfy the DVC.

Better: Used α-renamed copies for each t

β-Reduction

The most important **reduction rule** of the lambda calculus:

Definition

The (direct) (β)-reduction is defined as

$$(\beta) \qquad (\lambda x.s)\ t \xrightarrow{\beta} s[t/x]$$

If $r_1 \xrightarrow{\beta} r_2$, the we say r_1 directly reduces to r_2.

Contextual closure

The contextual closure of β-reduction is $\xrightarrow{C,\beta}$ defined as

$$C[s] \xrightarrow{C,\beta} C[t] \text{ iff } C \text{ is a context and } s \xrightarrow{\beta} t.$$

Examples

$$(\lambda x.x)\ (\lambda y.y) \xrightarrow{\beta} x[(\lambda y.y)/x] = \lambda y.y$$

$$(\lambda y.y\ y\ y)\ (x\ z) \xrightarrow{\beta} (y\ y\ y)[(x\ z)/y] = (x\ z)\ (x\ z)\ (x\ z)$$

To obey the DVC after a β-reduction: Apply α-renaming, e.g.

$$(\lambda x.(x\ x))\ (\lambda y.y) \xrightarrow{\beta} \underbrace{(\lambda y.y)\ (\lambda y.y)}_{\text{violates the DVC}} =_\alpha \underbrace{(\lambda y_1.y_1)\ (\lambda y_2.y_2)}_{\text{satisfies the DVC}}$$

Reduction Strategy

To evaluate an expression, β-reduction has to be applied to subexpressions, e.g.

$$((\lambda x.x)\ (\lambda y.y))\ (\lambda z.z) \xrightarrow{C,\beta} (\lambda y.y)\ (\lambda z.z)$$

Those subexpressions are called a **redex** (reducible expression).

Using $\xrightarrow{C,\beta}$ is not deterministic, e.g. $((\lambda x.x\ x)\ ((\lambda y.y)\ (\lambda z.z)))$ has two redexes:

- $((\lambda x.x\ x)\ ((\lambda y.y)\ (\lambda z.z))) \xrightarrow{C,\beta} ((\lambda y.y)\ (\lambda z.z))\ ((\lambda y.y)\ (\lambda z.z))$
- $((\lambda x.x\ x)\ ((\lambda y.y)\ (\lambda z.z))) \xrightarrow{C,\beta} ((\lambda x.x\ x)\ (\lambda z.z))$.

Fixing the position where to apply the reduction is called a **reduction strategy**.

We do it soon, but first we consider arbitrary $\xrightarrow{C,\beta}$-steps.

CHURCH-ROSSER-THEOREM

- The diamond property
- Confluence
- $\to_1$-reduction
- Proof of the Church-Rosser-Theorem

Notation

For a binary relation $\to \subseteq (M \times M)$, we denote with

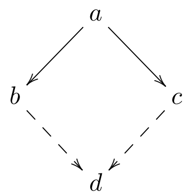
- $\leftrightarrow$ the symmetric closure of $\to$ (i.e. $a \leftrightarrow b$ iff $a \to b$ or $b \to a$).
- $\xrightarrow{i}$ the i-fold composition of $\to$
  ($a \xrightarrow{0} a$ for all $a \in M$ and for $i > 0$: $a \xrightarrow{i} b$, if $\exists b' \in M : a \to b'$ and $b' \xrightarrow{i-1} b$).
- $\xrightarrow{i \vee j}$ is the union of the i-fold and the j-fold composition
  ($a \xrightarrow{i \vee j} b$ iff $a \xrightarrow{i} b$ or $a \xrightarrow{j} b$).
  In particular, $\xrightarrow{0 \vee 1}$ is the reflexive-closure of $\to$.
- $\xrightarrow{*}$ the reflexive-transitive closure of $\to$ ($a \xrightarrow{*} b$ iff $\exists i \in \mathbb{N}_0 : a \xrightarrow{i} b$).
- $\xleftrightarrow{*}$ the reflexive-transitive closure of $\leftrightarrow$.
- $\xrightarrow{+}$ the transitive closure of $\to$ ($a \xrightarrow{+} b$ iff $\exists i \in \mathbb{N} : a \xrightarrow{i} b$).

The Diamond Property and Confluence

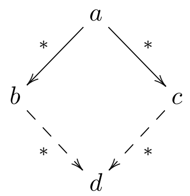
Definition

A binary relation $\to \subseteq M \times M$

- has the diamond property iff whenever $a \to b$ and $a \to c$ there exists $d \in M$ such that $b \to d$ and $c \to d$.
- is confluent iff $\xrightarrow{*}$ has the diamond property.



$\to$ has the diamond property



$\to$ is confluent

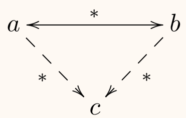
Motivation

- our goal is to show that $\xrightarrow{C,\beta}$ is confluent
- if confluence holds, then normal forms are unique: if we reduce all $\xrightarrow{C,\beta}$-redexes, then we get the same expression (up to α-renaming) independently from the order and positions where the reductions where applied

A Consequence of Confluence

Lemma 3.3.2

If reduction relation $\to$ is confluent, then $a \xleftrightarrow{*} b$ implies $\exists c : a \xrightarrow{*} c \wedge b \xrightarrow{*} c$

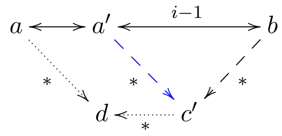


Proof. By induction on $a \xleftrightarrow{i} b$.

- Base case: if $i = 0$, then $a = b$ and the claim holds
- ...

Proof (Cont'd)

If $i > 0$, then $\exists a' : a \leftrightarrow a' \xleftrightarrow{i-1} b$. By induction hypothesis $\exists c' : a' \xrightarrow{*} c'$ and $b \xrightarrow{*} c'$.



- If $a \to a'$, then $a \to a' \xrightarrow{*} c'$ and thus $a \xrightarrow{*} c'$. Since also $b \xrightarrow{*} c'$, the claim holds.
- If $a' \to a$, then $a' \xrightarrow{*} a$.
  Since $\to$ is confluent, $\xrightarrow{*}$ has the diamond property and thus:
  from $a' \xrightarrow{*} a$ and $a' \xrightarrow{*} c'$, we obtain d with $a \xrightarrow{*} d$ and $c' \xrightarrow{*} d$.
  Since $b \xrightarrow{*} c' \xrightarrow{*} d$, the claim holds. □

Diamond Property: Inheritance from $\to$ to $\xrightarrow{*}$

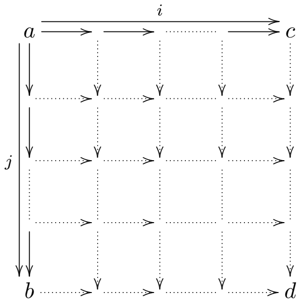
Lemma 3.3.3

Let $\to$ be a binary relation and $\xrightarrow{*}$ be its reflexive-transitive closure.
If $\to$ has the diamond property, then $\xrightarrow{*}$ has the diamond property.

Proof: By induction on (i, j) one can show that:

If $a \xrightarrow{i} b$ and $a \xrightarrow{j} c$ then $\exists d$: $b \xrightarrow{i} d$ and $c \xrightarrow{j} d$.

The inner square diagrams follow from the diamond property of $\to$.

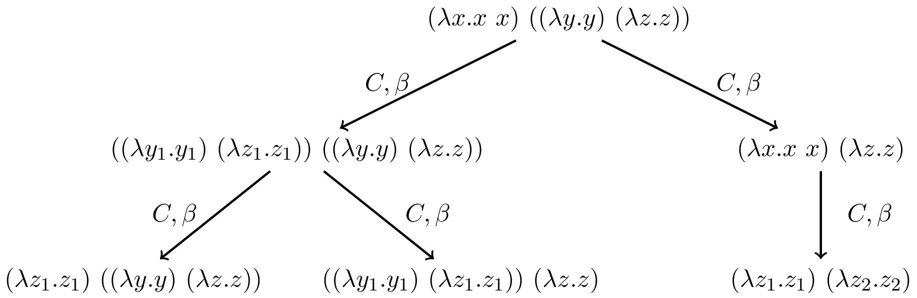


Closures of $\xrightarrow{\beta}$

- With $s \xleftrightarrow{C,\beta} t$ we denote the symmetric closure of $\xrightarrow{C,\beta}$ (i.e. $s \xleftrightarrow{C,\beta} t$ iff $s \xrightarrow{C,\beta} t$ or $t \xrightarrow{C,\beta} s$)
- With $s \xrightarrow{C,\beta,*} t$ we denote the reflexive-transitive closure of $\xrightarrow{C,\beta}$
- With $s \xleftrightarrow{C,\beta,*} t$ we denote the reflexive-transitive closure of $\xleftrightarrow{C,\beta}$

The relation $\xleftrightarrow{C,\beta,*}$ is sometimes also called β-equivalence or also convertibility.

$\xrightarrow{C,\beta}$ does not have the diamond property



→ Confluence of $\xrightarrow{\beta}$ cannot be proved by the previous lemma

→ Idea:

- Another reduction relation $\to_1$ with $\xrightarrow{C,\beta,0\vee 1} \subseteq \to_1 \subseteq \xrightarrow{C,\beta,*}$ and $\xrightarrow{*}_1 = \xrightarrow{C,\beta,*}$
- Prove diamond-property of $\to_1$. This implies diamond-property of $\xrightarrow{*}_1 = \xrightarrow{C,\beta,*}$

$\to_1$-Reduction

Definition (Parallel Reduction $\to_1$)

The relation $\to_1 \subseteq (\mathbf{Expr} \times \mathbf{Expr})$ is inductively defined by:

1. $s \to_1 s$ for all expressions s.
2. if $s_1 \to_1 s_2$ and $t_1 \to_1 t_2$, then $(s_1\ s_2) \to_1 (t_1\ t_2)$.
3. if $s_1 \to_1 s_2$ and $t_1 \to_1 t_2$, then $((\lambda x.s_1)\ t_1) \to_1 s_2[t_2/x]$.
4. if $s \to_1 t$, then $\lambda x.s \to_1 \lambda x.t$.

Lemma 3.3.7

$\xrightarrow{C,\beta} \subseteq \to_1$

Proof. Let $C[(\lambda x.s)\ t] \xrightarrow{C,\beta} C[s[t/x]]$. We show $C[(\lambda x.s)\ t] \to_1 C[s[t/x]]$ by structural induction on C.

If $C = [\cdot]$, then $(\lambda x.s)\ t \to_1 s[t/x]$ by ❸, since $s \to_1 s$ and $t \to_1 t$ by ❶.

If $C \neq [\cdot]$ we use as IH that $C'[(\lambda x.s)\ t] \to_1 C'[s[t/x]]$ where C' is a proper subcontext of context C.

- If $C = (C'\ r)$, then $r \to_1 r$ by ❶, $C'[(\lambda x.s)\ t] \to_1 C'[s[t/x]]$ by the IH and thus $C[(\lambda x.s)\ t] = (C'[(\lambda x.s)\ t]\ r) \to_1 (C'[s[t/x]]\ r) = C[s[t/x]]$ by ❷.
- The case $C = (r\ C')$ is completely analogous to the previous one.
- If $C = \lambda y.C'$, then $C'[(\lambda x.s)\ t] \to_1 C'[s[t/x]]$ by the IH and thus $C[(\lambda x.s)\ t] = \lambda y.C'[(\lambda x.s)\ t] \to_1 \lambda y.C'[s[t/x]] = C[s[t/x]]$ by ❹. □

Examples for $\to_1$

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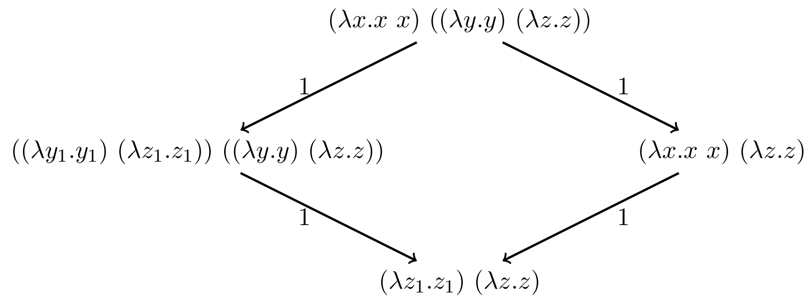

Examples for $\to_1$ (2)

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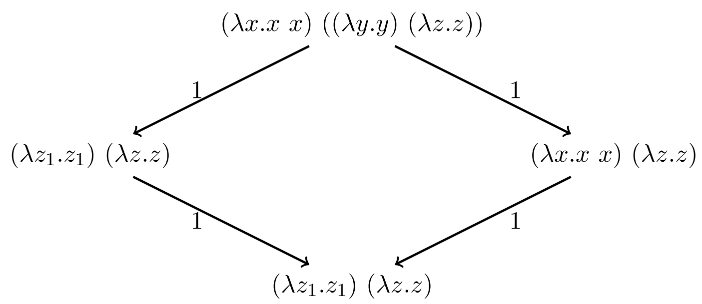

$\to_1$ and Substitutions

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Lemma 3.3.10

If $s \to_1 r$ and $t \to_1 u$ then $s[t/x] \to_1 r[u/x]$.

Proof. This can be shown by induction on $s \to_1 r$.
The base case $s = r$ can be shown by structural induction on s.
For the induction step, we only show one exemplary case:

- If $s = (s_1\ s_2) \to_1 (r_1\ r_2) = r$ with $s_i \to_1 r_i$ for $i = 1, 2$, the IH shows $s_i[t/x] \to_1 r_i[u/x]$ for $i = 1, 2$ and thus $(s_1[t/x]\ s_2[t/x]) \to_1 (r_1[u/x]\ r_2[u/x])$. Since $s[t/x] = (s_1\ s_2)[t/x]$ and $(r_1\ r_2)[u/x] = r[u/x]$, the claim holds.

Diamond Property for $\to_1$

Lemma 3.3.11

Relation $\to_1$ has the diamond property.

Proof. We show that whenever $s \to_1 t$ then for all r with $s \to_1 r$ there exists r' with $t \to_1 r'$ and $r \to r'$.

We use induction on the definition of $\to_1$ in $s \to_1 t$.

Base case: $t = s$, i.e. $s \to_1 s$. Then choose $r' = r$ and the claim holds.

For the induction step, all other cases of the definition of $\to_1$ have to be considered. We show one exemplary case.

Diamond Property for $\to_1$ (Cont'd)

If $s = ((\lambda x.s_1)\ s_2)$ and $t = t_1[t_2/x]$ where $s_i \to_1 t_i$, then for $s \to_1 r$ there are two cases:

- $s = ((\lambda x.s_1)\ s_2) \to_1 ((\lambda x.r_1)\ r_2)$ with $s_1 \to_1 r_1$ and $s_2 \to_1 r_2$.
  Applying the IH to $s_1 \to_1 t_1$ and $s_1 \to_1 r_1$ and also to $s_2 \to_1 t_2$ and $s_2 \to_1 r_2$ shows that there exists r_1' and r_2' such that: $t_i \to_1 r_i'$, $r_i \to_1 r_i'$ for $i = 1, 2$.
  This shows that $t_1[t_2/x] \to_1 r_1'[r_2'/x]$ and $((\lambda x.r_1)\ r_2) \to_1 r_1'[r_2'/x]$ (using the previous lemma). Thus the diamond property holds.
- If $s = ((\lambda x.s_1)\ s_2)$ and $r = r_1[r_2/x]$ where $s_1 \to_1 r_1$ and $s_2 \to_1 r_2$. Applying the IH to $s_1 \to_1 t_1$ and $s_1 \to_1 r_1$ and also to $s_2 \to_1 t_2$ and $s_2 \to_1 r_2$ shows that there exists r_1' and r_2' such that: $t_i \to_1 r_i'$, $r_i \to_1 r_i'$ for $i = 1, 2$. This shows that $t_1[t_2/x] \to_1 r_1'[r_2'/x]$ and $r_1[r_2/x] \to_1 r_1'[r_2'/x]$ (using the previous lemma). Thus the diamond property holds.

Coincidence of $\xrightarrow{C,\beta,*}$ and $\xrightarrow{*}_1$

Lemma 3.3.12

$\xrightarrow{C,\beta,*} = \xrightarrow{*}_1$

Proof (Sketch).
Since $\xrightarrow{C,\beta} \subseteq \to_1$ (Lemma 3.3.7), $\xrightarrow{C,\beta,*} \subseteq \xrightarrow{*}_1$ also holds.

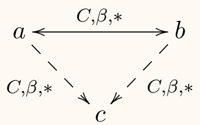
$\to_1 \subseteq \xrightarrow{\beta,*}$ can be proved by inspecting the different cases of the inductive definition of $\to_1$.

Finally, $\xrightarrow{*}_1 \subseteq \xrightarrow{C,\beta}$ holds, since $\xrightarrow{*}_1 \subseteq \left(\xrightarrow{C,\beta,*}\right)^*$ and $\xrightarrow{C,\beta,*}{}^* = \xrightarrow{C,\beta,*}$.

Church-Rosser-Theorem

Church-Rosser-Theorem

For the lambda calculus the following holds: If $a \xleftrightarrow{C,\beta,*} b$, then there exists c, such that $a \xrightarrow{C,\beta,*} c$ and $b \xrightarrow{C,\beta,*} c$



Proof.
Applying Lemma 3.3.3 for $\to_1$ (using Lemma 3.3.11) shows that $\to_1$ is confluent and that $\xrightarrow{*}_1$ has the diamond property.
With the equation of Lemma 3.3.12, we have that $\xrightarrow{C,\beta,*}$ has the diamond property and thus $\xrightarrow{C,\beta}$ is confluent. Finally, Lemma 3.3.2 then shows the claim. □

CALL-BY-NAME EVALUATION

- Reduction contexts
- Alternative definition with labeling
- Convergence
- Standardisation-Theorem

Call-by-Name Evaluation

Ideas:

- do not reduce below λ
- reduce the leftmost-outermost β-redex
- in $(\lambda x.s)\ t$ pass t to the function body without evaluating t

Definition

Reduction contexts R are built by the following grammar:

$$\mathbf{RCtxt} ::= [\cdot] \mid (\mathbf{RCtxt}\ \mathbf{Expr})$$

If $r_1 \xrightarrow{\beta} r_2$ and R is a reduction context,
then $R[r_1] \xrightarrow{name} R[r_2]$ is a call-by-name reduction step.

Example

$$
\begin{array}{ll}
 & (\lambda w.w)\ ((\lambda u.u)\ (\lambda v.v))\ (\lambda x.((\lambda y.y)\ (\lambda z.z))) \\
\xrightarrow{name} & (\lambda u.u)\ (\lambda v'.v')\ (\lambda x.((\lambda y.y)\ (\lambda z.z))) \\
\xrightarrow{name} & (\lambda v.v)\ (\lambda x.((\lambda y.y)\ (\lambda z.z))) \\
\xrightarrow{name} & \lambda x.((\lambda y.y)\ (\lambda z.z)) \\
\not\xrightarrow{name} &
\end{array}
$$

Alternative Definition of Call-by-Name Reduction

Use a labeling algorithm to mark the redex:

- For expression s, start with $s^\star$.
- Apply the label shifting as long as possible:

$$(s_1\ s_2)^\star \Rightarrow (s_1^\star\ s_2)$$

The result is of the form $(s_1^\star\ s_2\ \dots\ s_n)$, where s_1 is not an application.

- If s_1 is an abstraction $\lambda x.s_1'$ and $n \geq 2$, then reduce as follows:

$$(\lambda x.s_1')\ s_2\ \dots s_n \xrightarrow{name} (s_1'[s_2/x]\ \dots s_n)$$

- If s_1 is an abstraction and $n = 1$, then no call-by-name reduction is applicable (since the whole expression is an abstraction)
- If s_1 is a variable, then no call-by-name reduction is applicable (since a free variable has been detected)

Example

$$\left(\left((\lambda w.w)^\star\ ((\lambda u.u)\ (\lambda v.v))\right)^\star\ (\lambda x.((\lambda y.y)\ (\lambda z.z)))\right)^\star$$
$$\xrightarrow{name} \left(((\lambda u.u)^\star\ (\lambda v.v))^\star\ (\lambda x.((\lambda y.y)\ (\lambda z.z)))\right)^\star$$
$$\xrightarrow{name} \left((\lambda v.v)^\star\ (\lambda x.((\lambda y.y)\ (\lambda z.z)))\right)^\star$$
$$\xrightarrow{name} (\lambda x.((\lambda y.y)\ (\lambda z.z)))^\star$$

Convergence

- Call-by-name reduction is deterministic: if $s \xrightarrow{name} t$ and $s \xrightarrow{name} t' \implies t = t'$
- No call-by-name reducion is applicable iff $s = R[x]$ or if s is an abstraction (called an FWHNF (functional weak head normal form))
- Reaching an FWHNF means success
- $\xrightarrow{name,+}$ and $\xrightarrow{name,*}$ are the transitive and reflexive-transitive closure of $\xrightarrow{name}$.

Definition

Expression s (call-by-name) converges $s{\downarrow}$ iff $\exists$abstraction $v : s \xrightarrow{name,*} v$.

If s does not converge, we write $s{\Uparrow}$ and say s diverges.

Standardisation (Preparations)

- Goal: show that call-by-name evaluation is optimal w.r.t. convergence
- Technique: $\to_1$-reduction, also on contexts:
  C is treated like an expression with constant $[\cdot]$
- If $R \to_1 R'$ for a reduction context R, then R' is also a reduction context.
- If $C \to_1 C'$, $s \to_1 s'$, where C is a context, then $C[s] \to_1 C[s']$

Definition ($\xrightarrow{name}_1$ and $\xrightarrow{int}_1$)

If $R \to_1 R'$, $s \to_1 s'$, $t \to_1 t'$, and R is a reduction context, then

$$R[(\lambda x.s)\ t] \xrightarrow{name}_1 R'[s'[t'/x]]$$

Let $\xrightarrow{int}_1 := \to_1 \setminus \xrightarrow{name}_1$ be the internal $\xrightarrow{1}$-reduction.

Note that: $\xrightarrow{name} \subset \xrightarrow{name}_1 \subset \to_1$

Standardisation (Preparations, cont'd)

Counting the contracted redexes:

Definition

Define the measure $\phi : \to_1 \to \mathbb{N}_0$ inductively as

$$\begin{aligned}
\phi(x \to_1 x) &= 0, \text{if } x \text{ is a variable}\\
\phi(\lambda x.s \to_1 \lambda x.s') &= \phi(s \to s')\\
\phi(((\lambda x.s)\ t) \to_1 s'[t'/x]) &= 1 + \phi(s \to_1 s') + k \cdot \phi(t \to_1 t'), \text{where } k \text{ is the number of free occurrences of } x \text{ in } s\\
\phi((s\ t) \to_1 (s'\ t')) &= \phi(s \to_1 s') + \phi(t \to_1 t')
\end{aligned}$$

Measure ϕ is defined for every $\to_1$-step and it is well-founded.

Splitting $\to_1$

Lemma 3.4.10

If $s \to_1 t$, then $s \xrightarrow{name,*} s' \xrightarrow{int}_1 t$.

Proof. If $s \xrightarrow{int}_1 t$, then the claim holds.
Otherwise, $s \xrightarrow{name}_1 t$, i.e. $s = R[(\lambda x.r)\ u]$, $r \to_1 r'$, $u \to_1 u'$, $R \to_1 R'$, $t = R'[r'[u'/x]]$.
Then $s \xrightarrow{name} R[r[u/x]] \to_1 R'[r'[u'/x]]$ and this can be iterated. If the iteration stops, the demanded reduction sequence is constructed.
For termination, we verify $\phi(s \to_1 t) > \phi(R[r[u/x]] \to_1 R'[r'[u'/x]])$:

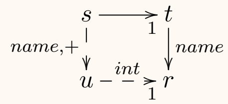
$$\begin{aligned}\phi(R[(\lambda x.r)\ u] \to_1 R'[r'[u'/x]]) &= \phi(R \to_1 R') + \phi((\lambda x.r)\ u \to_1 r'[u'/x]) \\ &= \phi(R \to_1 R') + 1 + \phi(r \to_1 r') + k\phi(u \to_1 u') \\ \phi(R[r[u/x]] \to_1 R'[r'[u'/x]]) &= \phi(R \to_1 R') + \phi(r \to_1 r') + k\phi(u \to_1 u')\end{aligned}$$

where k is the number of free occurrences of x in r. □

Shifting $\xrightarrow{name}$ over $\to_1$

Lemma 3.4.11

Let $s \to_1 t \xrightarrow{name} r$, then there exists u such that $s \xrightarrow{name,+} u \xrightarrow{int}_1 r$.



Proof.

By Lemma 3.4.10 $s \to_1 t \xrightarrow{name} r$ can be written as $s \xrightarrow{name,*} t' \xrightarrow{int}_1 t \xrightarrow{name} r$.

Since $t \xrightarrow{name} r$, we can assume that $t = R[(\lambda x.t_0)\ t_1] \xrightarrow{name} R[t_0[t_1/x]] = r$.

Since $t' \xrightarrow{int}_1 t$ is internal, $t' = R'[(\lambda x.t_0')\ t_1']$ where $R' \to_1 R$, $t_0' \to_1 t_0$, and $t_1' \to_1 t_1$.

Then $t' = R'[(\lambda x.t_0')\ t_1'] \xrightarrow{name} R'[t_0'[t_1'/x]] \xrightarrow{int}_1 R[t_0[t_1/x]] = r$ holds. □

Shifting $\xrightarrow{name,*}$ over $\to_1$

Applying Lemma 3.4.11 iteratively shows:

Lemma 3.4.12

If $s \to_1 t \xrightarrow{name,*} v$, v is an FWHNF, then $s \xrightarrow{name,*} v' \xrightarrow{int}_1 v$ where v' is an FWHNF.

The induction is on the number of $\xrightarrow{name}$-steps.

For the base case, observe that $\xrightarrow{int}_1$ steps do not transform non-FWHNFs into FWHNFs.

For the induction step, apply Lemma 3.4.11 and use the induction hypothesis.

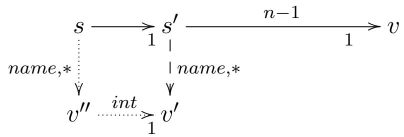
Standardisation-Theorem

Call-by-name evaluation is an optimal strategy w.r.t. termination:

Standardisation-Theorem

If $s \xrightarrow{C,\beta,*} v$ where v is a FWHNF, then $s\downarrow$.

Proof. The given sequence is also a sequence of $\to_1$-reductions, since $\xrightarrow{C,\beta} \subseteq \to_1$.
We show by induction on n: if $s \xrightarrow{n}_1 v$ where v is an FWHNF, then $s\downarrow$.
If $n = 0$, then the claim holds. The induction step:



Dashed steps follow from the IH, dotted steps follow by Lemma 3.4.12 □

CALL-BY-VALUE EVALUATION

Call-by-Value Evaluation

- Used in strict functional programming languages like ML, F$\#$, ...
- Difference to call-by-name: β-reduction is only permitted if the argument is a value (an abstraction)

Definition

The (direct) (β_{value})-reduction is defined as

$$(\lambda x.s)\ v \xrightarrow{\beta_{value}} s[v/x] \text{ where } v \text{ is a variable or an abstraction.}$$

We write $\xrightarrow{C,\beta_{value}}$ for the contextual closure of $\xrightarrow{\beta_{value}}$.

Call-by-Value Evaluation (Cont'd)

Call-by-value evaluation requires to evaluate parameters before calling the function!

Definition

Call-by-value reduction contexts E are built as follows:

$$\mathbf{ECtxt} ::= [\cdot] \mid (\mathbf{ECtxt}\ \mathbf{Expr}) \mid ((\lambda V.\mathbf{Expr})\ \mathbf{ECtxt})$$

If $r_1 \xrightarrow{\beta_{value}} r_2$ and E is a call-by-value reduction context, then

$$E[r_1] \xrightarrow{value} E[r_2]$$

is a call-by-value reduction.

Example

$$
\begin{array}{ll}
 & (\lambda x.(x\ (x\ x)))\ ((\lambda y.y\ y)\ (\lambda z.z)) \\
\xrightarrow{value} & (\lambda x.(x\ (x\ x)))\ ((\lambda z_1.z_1)(\lambda z_2.z_2)) \\
\xrightarrow{value} & (\lambda x.(x\ (x\ x)))\ (\lambda z_2.z_2) \\
\xrightarrow{value} & (\lambda z_2.z_2)\ ((\lambda z_3.z_3)\ (\lambda z_4.z_4)) \\
\xrightarrow{value} & (\lambda z_2.z_2)\ (\lambda z_4.z_4) \\
\xrightarrow{value} & (\lambda z_4.z_4)
\end{array}
$$

Alternative Definition with Labels

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- For s, start with $s^\star$.
- Exhaustively apply the rules:

$$(s_1\ s_2)^\star \Rightarrow (s_1^\star\ s_2)$$
$$(v^\star\ s) \Rightarrow (v\ s^\star) \qquad \text{if } v \text{ is an abstraction and } s \text{ is not an abstraction or a variable}$$

Three cases

- s is labeled with $\star$ and s is an abstraction: no reduction applicable
- $s = E[x^\star]$, i.e. a free variable in reduction position: no reduction applicable
- $s = E[(\lambda x.t)^\star\ v]$ where v is an abstraction or variable: Then $E[(\lambda x.t)\ v] \xrightarrow{value} E[t[v/x]]$

Example

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$$
\begin{aligned}
& ((\lambda x.(x\ (x\ x)))^\star\ ((\lambda y.y\ y)^\star\ (\lambda z.z))^\star)^\star \\
\xrightarrow{value}\ & ((\lambda x.(x\ (x\ x)))^\star\ ((\lambda z_1.z_1)^\star\ (\lambda z_2.z_2))^\star)^\star \\
\xrightarrow{value}\ & (\lambda x.(x\ (x\ x)))^\star\ (\lambda z_2.z_2))^\star \\
\xrightarrow{value}\ & \Big((\lambda z_3.z_3)^\star\ ((\lambda z_4.z_4)^\star\ (\lambda z_5.z_5))^\star\Big)^\star \\
\xrightarrow{value}\ & \Big((\lambda z_3.z_3)^\star\ (\lambda z_5.z_5)\Big)^\star \\
\xrightarrow{value}\ & (\lambda x_5.x_5)^\star
\end{aligned}
$$

Convergence

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Note that call-by-value reduction is deterministic.

Definition

Expression s converges for call-by-value evaluation:

$$s{\downarrow}_{value} \text{ iff } \exists \text{abstraction } v : s \xrightarrow{value,*} v.$$

If $\neg s{\downarrow}_{value}$, then we write $s{\Uparrow}_{value}$ (s diverges for call-by-value evaluation).

Call-by-Name vs. Call-by-Value

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Standardisation-Theorem immediately shows:

Corollary

For all expressions s: $s{\downarrow}_{value} \implies s{\downarrow}$

The converse does **not** hold:

- $\Omega := (\lambda x.x\ x)\ (\lambda x.x\ x)$.
- $\Omega \xrightarrow{name} \Omega$ and also $\Omega \xrightarrow{value} \Omega$
- $(\lambda x.\lambda y.y)\ \Omega \xrightarrow{name} \lambda y.y$, thus $(\lambda x.\lambda y.y)\ \Omega{\downarrow}$
- $(\lambda x.\lambda y.y)\ \Omega \xrightarrow{value} (\lambda x.\lambda y.y)\ \Omega$, thus $(\lambda x.\lambda y.y)\ \Omega{\Uparrow}_{value}$

Call-by-Name vs. Call-by-Value (Cont'd)

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Consider $f\ s_1\ s_2\ s_3$

- In call-by-value evaluation: first s_1, then s_2, then s_3, then the application
  → Evaluation order is predictable
- In call-by-name evaluation: first the application.
  When (if it all) s_1, s_2, s_3 are evaluated depends on the definition of f!
  → Evaluation order is not predictable
- This is relevant, if $s_i = \texttt{print}\ i$
- Main reason why strict functional languages (ML, Ocaml,...) permit direct side-effects, but non-strict functional languages (Haskell) forbid them.

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CALL-BY-NEED EVALUATION

Call-by-Need Evaluation

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- Also called: lazy evaluation with sharing
- It optimizes the call-by-name evaluation: by avoiding duplicated evaluations:
  - Consider $(\lambda x. \ldots\ x\ \ldots\ x)\ t$.
  - In call-by-name evaluation t is copied and perhaps evaluated several times!
  - Idea of call-by-need: share the result of evaluating t
  - We use a new construct for this sharing $\texttt{let}\ x = t\ \texttt{in}\ \ldots$

Expressions with `let`

$$\mathbf{Expr} ::= V \mid \lambda V.\mathbf{Expr} \mid (\mathbf{Expr}\ \mathbf{Expr}) \mid \texttt{let}\ V = \mathbf{Expr}\ \texttt{in}\ \mathbf{Expr}$$

Note that `let` in Haskell is recursive, here we have a non-recursive `let`.

Call-by-Need Lambda Calculus: Evaluation

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Reduction contexts are $\mathbf{R}_{need}$:

$$\begin{aligned}\mathbf{R}_{need} &::= \mathbf{LR}[\mathbf{A}] \mid \mathbf{LR}[\texttt{let}\ x = \mathbf{A}\ \texttt{in}\ \mathbf{R}_{need}[x]]\\ \mathbf{A} &::= [\cdot] \mid (\mathbf{A}\ \mathbf{Expr})\\ \mathbf{LR} &::= [\cdot] \mid \texttt{let}\ V = \mathbf{Expr}\ \texttt{in}\ \mathbf{LR}\end{aligned}$$

- $\mathbf{A}$ $\widehat{=}$ left into the application
- $\mathbf{LR}$ $\widehat{=}$ right into the `let`

Call-by-Need Lambda Calculus: Evaluation (Cont'd)

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Call-by-need reduction step $\xrightarrow{need}$, defined by:

$(lbeta)$ $R_{need}[(\lambda x.s)\ t] \xrightarrow{need} R_{need}[\texttt{let}\ x = t\ \texttt{in}\ s]$

(cp) $LR[\texttt{let}\ x = \lambda y.s\ \texttt{in}\ R_{need}[x]] \xrightarrow{need} LR[\texttt{let}\ x = \lambda y.s\ \texttt{in}\ R_{need}[\lambda y.s]]$

$(llet)$ $LR[\texttt{let}\ x = (\texttt{let}\ y = s\ \texttt{in}\ t)\ \texttt{in}\ R_{need}[x]]$
$\xrightarrow{need} LR[\texttt{let}\ y = s\ \texttt{in}\ (\texttt{let}\ x = t\ \texttt{in}\ R_{need}[x])]$

$(lapp)$ $R_{need}[(\texttt{let}\ x = s\ \texttt{in}\ t)\ r] \xrightarrow{need} R_{need}[\texttt{let}\ x = s\ \texttt{in}\ (t\ r)]$

- $(lbeta)$ and (cp) replace (β),
- $(lapp)$ and $(llet)$ adjust `let`s

Labeling-Algorithm

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- Labels: $\star, \diamond, \circledcirc$
- $\star \vee \diamond$ means $\star$ or $\diamond$
- For s, start with $s^\star$.

Shifting-Rules:

(1)	$(\texttt{let}\ x = s\ \texttt{in}\ t)^\star$	$\Rightarrow$	$(\texttt{let}\ x = s\ \texttt{in}\ t^\star)$
(2)	$(\texttt{let}\ x = C_1[y^\diamond]\ \texttt{in}\ C_2[x^\circledcirc])$	$\Rightarrow$	$(\texttt{let}\ x = C_1[y^\diamond]\ \texttt{in}\ C_2[x])$
(3)	$(\texttt{let}\ x = s\ \texttt{in}\ C[x^{\star\vee\diamond}])$	$\Rightarrow$	$(\texttt{let}\ x = s^\diamond\ \texttt{in}\ C[x^\circledcirc])$
(4)	$(s\ t)^{\star\vee\diamond}$	$\Rightarrow$	$(s^\diamond\ t)$

where (2) is preferred over (3)

Labeling-Algorithm: Reduction after Labeling

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$(lbeta)$ $((\lambda x.s)^\diamond\ t) \to \texttt{let}\ x = t\ \texttt{in}\ s$

(cp) $\texttt{let}\ x = (\lambda y.s)^\diamond\ \texttt{in}\ C[x^\circledcirc] \to \texttt{let}\ x = \lambda y.s\ \texttt{in}\ C[\lambda y.s]$

$(llet)$ $\texttt{let}\ x = (\texttt{let}\ y = s\ \texttt{in}\ t)^\diamond\ \texttt{in}\ C[x^\circledcirc]$
$\to \texttt{let}\ y = s\ \texttt{in}\ (\texttt{let}\ x = t\ \texttt{in}\ C[x])$

$(lapp)$ $((\texttt{let}\ x = s\ \texttt{in}\ t)^\diamond\ r) \to \texttt{let}\ x = s\ \texttt{in}\ (t\ r)$

Example

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$$
\begin{array}{ll}
 & (\texttt{let}\ x = (\lambda u.u)\ (\lambda w.w)\ \texttt{in}\ ((\lambda y.y)\ x))^\star \\
\xrightarrow{need,lbeta} & (\texttt{let}\ x = (\lambda u.u)\ (\lambda w.w)\ \texttt{in}\ (\texttt{let}\ y = x\ \texttt{in}\ y))^\star \\
\xrightarrow{need,lbeta} & (\texttt{let}\ x = (\texttt{let}\ u = \lambda w.w\ \texttt{in}\ u)\ \texttt{in}\ (\texttt{let}\ y = x\ \texttt{in}\ y))^\star \\
\xrightarrow{need,llet} & (\texttt{let}\ u = \lambda w.w\ \texttt{in}\ (\texttt{let}\ x = u\ \texttt{in}\ (\texttt{let}\ y = x\ \texttt{in}\ y)))^\star \\
\xrightarrow{need,cp} & (\texttt{let}\ u = (\lambda w.w)\ \texttt{in}\ (\texttt{let}\ x = (\lambda w.w)\ \texttt{in}\ (\texttt{let}\ y = x\ \texttt{in}\ y)))^\star \\
\xrightarrow{need,cp} & (\texttt{let}\ u = (\lambda w.w)\ \texttt{in}\ (\texttt{let}\ x = (\lambda w.w)\ \texttt{in}\ (\texttt{let}\ y = (\lambda w.w)\ \texttt{in}\ y)))^\star \\
\xrightarrow{need,cp} & (\texttt{let}\ u = (\lambda w.w)\ \texttt{in}\ (\texttt{let}\ x = (\lambda w.w)\ \texttt{in}\ (\texttt{let}\ y = (\lambda w.w)\ \texttt{in}\ (\lambda w.w))))
\end{array}
$$

- The final expression is a call-by-need FWHNF
- Call-by-need FWHNF: expression of the form $LR[\lambda x.s]$, i.e.

$\texttt{let}\ x_1 = s_1\ \texttt{in}$
$(\texttt{let}\ x_2 = s_2\ \texttt{in}$
$(\ldots$
$(\texttt{let}\ x_n = s_n\ \texttt{in}\ \lambda x.s)))$

Convergence

Definition

Expression s converges for call-by-need evaluation:

$$s{\downarrow}_{need} \iff \exists \text{ FWHNF } v : s \xrightarrow{need,*} v$$

Proposition

Let s be (`let`-free) expression, then $s{\downarrow} \iff s{\downarrow}_{need}$.

→ W.r.t. convergence call-by-name and call-by-need are the same

CONTEXTUAL EQUIVALENCE

Equality in the Lambda Calculus

Up to now, we used three notions of equality:

- Syntactic equality
- α-equivalence
- β-convertibility $\xrightarrow{C,\beta,*}$

All of them are quite restrictive

→ We introduce a semantic equivalence, called contextual equivalence

Idea of Contextual Equivalence

Leibniz' law of the identity of indiscernibles:

- if objects o_1 and o_2 have the same property for all properties, then o_1 is identical to o_2.
- equality thus means: in every context, we can exchange o_1 by o_2, but no difference is observable.

For program calculi like the lambda calculus:

Expressions s and t are equal iff
their behaviour (convergence) cannot be distinguished independently in which context ($\forall C : C[s]$ and $C[t]$ ...) they are used.

Contextual Approximation and Equivalence

Definition

For the call-by-name lambda calculus:
contextual approximation $\leq_c$ and contextual equivalence $\sim_c$ are defined as

- $s \leq_c t$ iff $\forall C : C[s]{\downarrow} \implies C[t]{\downarrow}$
- $s \sim_c t$ iff $s \leq_c t$ und $t \leq_c s$

For the call-by-value lambda calculus:
contextual approximation $\leq_{c,value}$ and contextual equivalence $\sim_{c,value}$ are defined as:

- $s \leq_{c,value} t$ iff $\forall C$: If $C[s], C[t]$ are closed and $C[s]{\downarrow_{value}}$, then also $C[t]{\downarrow_{value}}$
- $s \sim_{c,value} t$ iff $s \leq_{c,value} t$ and $t \leq_{c,value} s$

We omit the call-by-need lambda calculus.

Closing vs. Non-Closing Contexts

- call-by-name: no difference if all, or only closing contexts are used in $\sim_c$
- call-by-value: there is a difference:
  $x \sim_{c,value} \lambda y.(x\ y)$ holds, but would not hold, if all contexts are used

	Variables represent
call-by-name	any expression
call-by-value	any value

Remark:

- the transformation $s \to \lambda x.(s\ x)$ is called eta-expansion
- the inverse transformation $\lambda x.(s\ x) \to s$ is called eta-reduction

Properties of Contextual Equivalence

Contextual equivalence is the coarsest equivalence that distinguishes obviously different expressions.
An important property is:

Proposition

- $\sim_c$ and $\sim_{c,value}$ are congruences, i.e. they are equivalence relations (i.e. reflexive, symmetric & transitive) and compatible with contexts ($s \sim t \implies C[s] \sim C[t]$).
- $\leq_c$ and $\leq_{c,value}$ are precongruences, i.e. they are preorders (i.e. reflexive & transitive) and compatible with contexts ($s \leq t \implies C[s] \leq C[t]$).

Proof: We only consider the precongruences, since the congruences follows by symmetry.

(next slide)

$\leq_c$ is a Precongruence

- reflexivity: for all contexts and expressions $C[s]{\downarrow} \implies C[s]{\downarrow}$, thus $s \leq_c s$
- transitivity: Let $r \leq_c s$ and $s \leq_c t$ and $C[r]{\downarrow}$.
  We have to show $C[t]{\downarrow}$:
  From $r \leq_c s$ we have $C[s]{\downarrow}$.
  From $s \leq_c t$ we also have $C[t]{\downarrow}$
- compatibility: Let $s \leq_c t$ and C be a context.
  We have to show $C[s] \leq_c C[t]$
  Let C' be a context such that $C'[C[s]]{\downarrow}$.
  Since $C'[C[\cdot]]$ is also a context, $s \leq_c t$ shows $C'[C[t]]{\downarrow}$

$\leq_{c,value}$ is a Precongruence

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- reflexivity and compatibility: similar to $\leq_c$
- transitivity: let $r \leq_{c,value} s$ and $s \leq_{c,value} t$

Let C be a context such that $C[r]$ and $C[t]$ are closed and $C[r]{\downarrow_{value}}$.

We have to show $C[t]{\downarrow_{value}}$. If $C[s]$ is also closed, the reasoning is as for $\leq_c$.

Otherwise, assume $FV(C[s]) = \{x_1, \ldots, x_n\}$.
Let $v_1, \ldots, v_n$ be arbitrary closed values and $D = (\lambda x_1, \ldots, x_n.[\cdot])\ v_1\ \ldots\ v_n$.
Since $C[r]$ and $C[t]$ are closed:

- $D[C[r]] \xrightarrow{value,*} C[r]$ and $D[C[t]] \xrightarrow{value,*} C[t]$
- Thus: $D[C[r]]{\downarrow_{value}} \iff C[r]{\downarrow_{value}}$ and $D[C[t]]{\downarrow_{value}} \iff C[t]{\downarrow_{value}}$

Since $C[r]{\downarrow_{value}}$, we have $D[C[r]]{\downarrow_{value}}$.
From $r \leq_{c,value} s$, we have $D[C[s]]{\downarrow}$.
From $s \leq_{c,value} t$, we have $D[C[t]]{\downarrow_{value}}$ and thus also $C[t]{\downarrow_{value}}$. □

Program Transformations

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- Program transformation $s \to t$ is correct iff $s \sim_c t$ holds
- Congruence property shows that local transformations preserve "global" equivalence: if $s \sim_c t$ then $C[s] \sim_c C[t]$
- Proving correctness is usually hard, because of the universal quantification on all contexts
- Decision problems $s \overset{?}{\sim}_c t$ or $s \overset{?}{\not\sim}_c t$ are undecidable, since:
  $s \overset{?}{\not\sim}_c \Omega$ can be used to encode the halting problem
  (and since the lambda calculus is Turing complete)

(β) is Correct in the Call-by-Name Lambda Calculus

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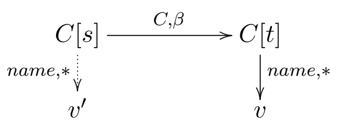
Proposition

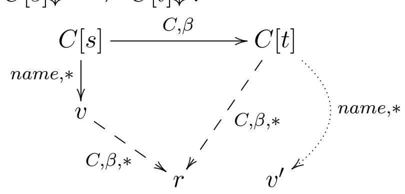
In the call-by-name lambda calculus: if $s \xrightarrow{\beta} t$, then $s \sim_c t$.

Proof. Let $s \xrightarrow{\beta} t$ and C be a context.

$C[t]{\downarrow} \implies C[s]{\downarrow}$:



$C[s]{\downarrow} \implies C[t]{\downarrow}$:



⟶ given reductions — – ➤ follows from the Church-Rosser-Theorem
v, r, v' are WHNFs. ⋯➤ follows from the Standardisation-Theorem

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CONTEXT LEMMA

Context Lemma: Ideas and Preparations

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- In general, a context lemma states that
  it suffices to check a subset of all contexts to conclude contextual equivalence
- We only consider the case of the call-by-name lambda calculus
- We require multi-contexts: Contexts with several (or no) holes

  We write $M[\cdot_1, \ldots, \cdot_n]$ for a multi-context with n holes

  We write $M[s_1, \ldots, s_n]$ if the hole $\cdot_i$ of M is replaced by s_i (for $i = 1, \ldots, n$).

Context Lemma

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Context Lemma

Let s and t be closed expressions. If for all reductions contexts R, the implication $R[s]{\downarrow} \implies R[t]{\downarrow}$ holds, then also $s \leq_c t$ holds.

Proof. We prove the more general claim using multi-contexts:

> If for all closed expressions s_i, t_i and for $i = 1, \ldots, n$: for all reduction contexts R the implication $R[s_i]{\downarrow} \implies R[t_i]{\downarrow}$ holds, then for all multi-contexts M the implication $M[s_1, \ldots, s_n]{\downarrow} \implies M[t_1, \ldots, t_n]{\downarrow}$ holds.

The context lemma follows with $n = 1$.

Proof of the Context Lemma

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> If for all closed expressions s_i, t_i and for $i = 1, \ldots, n$: for all reduction contexts R the implication $R[s_i]{\downarrow} \implies R[t_i]{\downarrow}$ holds, then for all multi-contexts M the implication $M[s_1, \ldots, s_n]{\downarrow} \implies M[t_1, \ldots, t_n]{\downarrow}$ holds.

Assume, the preconditions hold and $M[s_1, \ldots, s_n] \xrightarrow{name,m} v$ where v is a WHNF.
We use induction on the following pair, ordered lexicographically;

1. The number m of call-by-name reductions from $M[s_1, \ldots, s_n]$ to a WHNF.
2. The number n of holes of M.

Base case:

- Let $n = 0$ and m arbitrary: Then M is an expression and the claim holds.

Induction step:

- Let $n > 0$, i.e. M has holes
- We split into two cases (next slide)

Proof of the Context Lemma (Cont'd)

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Case: There exists a hole i, s.t. $M[s_1, \ldots, s_{i-1}, \cdot_i, s_{i+1}, \ldots, s_n]$ is a reduction context

- Then there exists a hole j, such that $M[r_1, \ldots, r_{j-1}, \cdot_j, r_{j+1}, \ldots, r_n]$ is a reduction context for any expressions $r_1, \ldots, r_n$.
- Let $M' = M[\cdot_1, \ldots, \cdot_{j-1}, s_j, \cdot_{j+1}, \ldots, \cdot_n]$
- Since $M'[s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n] = M[s_1, \ldots, s_n]$, they have the same call-by-name evaluation
- M' has $n-1$ holes, i.e. we can apply the IH, showing $M'[t_1, \ldots, t_{j-1}, t_{j+1}, t_n]{\downarrow}$.
- $C_s = M[s_1, \ldots, s_{j-1}, \cdot_j, s_{j+1}, \ldots, s_n]$ and $C_t = M[t_1, \ldots, t_{j-1}, \cdot_j, t_{j+1}, \ldots, t_n]$ are both reduction contexts, $M'[t_1, \ldots, t_{j-1}, t_{j+1}, t_n] = C_t[s_j]$ and $C_t[s_j]{\downarrow}$. Thus the precondition shows that $C_t[t_j]{\downarrow}$.
- Since $C_t[t_j] = M[t_1, \ldots, t_n]$, this shows the claim.

Proof of the Context Lemma (Cont'd)

Case $n > 0$ and no hole is a reduction context.

- If $m = 0$, then $M[s_1, \ldots, s_n]$ is a WHNF and $M[t_1, \ldots, s_n]$ must be a WHNF too.
- Otherwise, $M[s_1, \ldots, s_n] \xrightarrow{name} s' \xrightarrow{name,m-1} v$
- Inspect what can happen with the subexpressions $s_1, \ldots, s_n$ in M
- Since no hole of M is in a reduction context they can only change their position and maybe duplicated or removed.
- Since $s_1, \ldots, s_n$ are closed no other expression can be copied inside any s_i.
- Thus: There exists M' with k holes, such that
  - $s' = M'[s_{f(1)}, \ldots, s_{f(m)}]$ where $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$.
  - $M[r_1, \ldots, r_n] \xrightarrow{name} M'[r_{f(1)}, \ldots, r_{f(m)}]$ for any expressions $r_1, \ldots, r_n$
  - in particular, $M[t_1, \ldots, t_n] \xrightarrow{name} M'[t_{f(1)}, \ldots, r_{t(m)}] = t'$.

Since $s' \xrightarrow{name,m-1} v$ and the precondition holds for all pairs $s_{f(i)}, t_{f(i)}$ for $i = 1, \ldots, m$ we can apply the IH to s' and t' showing $t'{\downarrow}$ and thus also $t{\downarrow}$. □

Equivalences on Open Expressions

Proposition 3.8.2

Let s and t be expressions with free variables $x_1, \ldots, x_n$ Then $s \leq_c t$ iff for all closed expressions $t_1, \ldots, t_n$: $s[t_1/x_1, \ldots, t_n/x_n] \leq_c t[t_1/x_1, \ldots, t_n/x_n]$

Proof.

- "$\Rightarrow$": This follows since $\leq_c$ is a precongruence and since (β) is correct.
- "$\Leftarrow$": This can be shown by the context lemma and an induction on the number of variables.

Contextual Equivalence and Call-by-Value

In the call-by-value lambda calculus:

- $(\beta_{value}) \subseteq \sim_{c,value}$ (without a proof)
- $(\beta) \not\subseteq \sim_{c,value}$, since $((\lambda x.(\lambda y.y))\ \Omega){\Uparrow_{value}}$ and $\lambda y.y{\downarrow_{value}}$

The contextual equivalences w.r.t. call-by-name and call-by-value evaluation are not related:

$$\sim_c \not\subseteq \sim_{c,value} \text{ and } \sim_{c,value} \not\subseteq \sim_c$$

E.g. $((\lambda x.(\lambda y.y))\ \Omega)\ \sim_{c,value}\ \Omega$ but $((\lambda x.(\lambda y.y))\ \Omega)\ \not\sim_c\ \Omega$.

Least and Greatest Elements

Proposition

All closed diverging expressions are least elements w.r.t. $\leq_c$ and $\leq_{c,value}$.
For instance $\Omega \leq_c s$ and also $\Omega \leq_{c,value} s$ for all expressions s.

With $K := \lambda x.\lambda y.x$, $Y := \lambda f.(\lambda x.(f\ (x\ x)))\ (\lambda x.(f\ (x\ x)))$, and $Z := \lambda f.(\lambda x.(f\ \lambda z.(x\ x)\ z))\ (\lambda x.(f\ \lambda z.(x\ x)\ z))$:

- $Y\ K$ is a greatest element of $\leq_c$, i.e. $\forall s : s \leq_c Y\ K$
- $Z\ K$ is a greatest element of $\leq_{c,value}$, i.e. $\forall s : \leq_c Z\ K$

eteness

We only prove the proposition for the call-by-name lambda calculus.

Least Elements w.r.t. $\leq_c$

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- Let $\bot$ be a closed diverging expression and s be an arbitrary closed expression.
- Let R be an arbitrary reduction context, then $R[\bot]$ cannot converge, i.e. $R[\bot]\Uparrow$.
- The context lemma now immediately shows $\bot \leq_c s$.
- Since $\bot$ is closed, Proposition 3.8.2 shows $\bot \leq_c s$ for any (perhaps also open) expression s

Greatest Elements w.r.t. $\leq_c$

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- $Y := \lambda f.(\lambda x.(f\ (x\ x)))\ (\lambda x.(f\ (x\ x)))$
- Let $r_y = (\lambda x.K\ (x\ x))$.
- Then $Y\ K \xrightarrow{C,\beta} r_y\ r_y \xrightarrow{C,\beta} K\ (r_y\ r_y)$
- $(Y\ K)\ s_1\ \ldots\ s_n \xrightarrow{C,\beta,*} K\ (r_y\ r_y)\ s_1\ \ldots\ s_n \xrightarrow{C,\beta,*} (r_y\ r_y)\ s_2\ \ldots\ s_n \xrightarrow{C,\beta,*} (r_y\ r_y) \xrightarrow{C,\beta} K\ (r_y\ r_y) \xrightarrow{C,\beta} \lambda x.(r_y\ r_y)$
- The Standardisation-Theorem shows that for all R: $R[(Y\ K)]\downarrow$.
- Since $(Y\ K)$ is closed, the context lemma shows $s \leq_c (Y\ K)$ for every closed expression s
- Proposition 3.8.2 shows $s \leq_c (Y\ K)$ for any (perhaps also open) expression s.

The Z-Combinator and Call-By-Value

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We explain the call-by-value evaluation of $(Z\ K)$:

- $Z = \lambda f.(\lambda x.(f\ \lambda z.(x\ x)\ z))\ (\lambda x.(f\ \lambda z.(x\ x)\ z))$
- Let $r_z = (\lambda x.(K\ \lambda z.(x\ x)\ z))$
- $Z\ K \xrightarrow{value} r_z\ r_z \xrightarrow{value} K\ \lambda z.((r_z\ r_z)\ z) \xrightarrow{value} \lambda y.\lambda z.(r_z\ r_z)\ z$
- For values $v_1,\ldots,v_n$: $(Z\ K)\ v_1\ \ldots\ v_n \xrightarrow{value,*} (r_z\ r_z)\ v_1\ \ldots\ v_n \xrightarrow{value,*} (\lambda y.\lambda z.(r_z\ r_z)\ z)\ v_1\ \ldots\ v_n \xrightarrow{value} (\lambda z.(r_z\ r_z)\ z)\ v_2 \ldots\ v_n \xrightarrow{value} (r_z\ r_z)\ v_2\ v_3 \ldots\ v_n \xrightarrow{value,*} (r_z\ r_z) \xrightarrow{value,*} \lambda y.\lambda z.(r_z\ r_z)\ z$

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TURING COMPLETENESS OF THE LAMBDA CALCULUS

Turing Completeness

- We do not provide a formal proof
- In the lecture notes, a Haskell-implementation of Turing machines can be found
- We argue that the program constructs used in the program, can be encoded in the lambda calculus

Constructs:

- Named functions
- Recursion
- Data (booleans, numbers, pairs, lists)

Function Definitions

Non-recursive Haskell-function $f\ x_1\ \dots\ x_n = e$ can be represented by $\lambda x_1, \dots, x_n.e$

Recursive functions can be encoded by the fixpoint combinator:

For simplicity, let us assume, that e only calls f, but no other functions.

Then f can be encoded by $Y\ (\lambda f.\lambda x_1\ \dots\ x_n.e)$:

- Let $F = (\lambda f.\lambda x_1\ \dots\ x_n.e)$ and $r_y = (\lambda x.F\ (x\ x))$.
- Then $Y\ F \xrightarrow{C,\beta} r_y\ r_y = (\lambda x.F\ (x\ x))\ r_y \xrightarrow{C,\beta} F\ (r_y\ r_y) \xleftarrow{C,\beta} F\ (Y\ F)$, i.e. $Y\ F \sim_c F\ (Y\ F)$.
- Thus $Y\ F \sim_c F^i\ (Y\ F)$ where F^i is the i-fold application of F and also $Y\ F \sim_c F\ (Y\ F) \sim_c \lambda x_1, \dots, x_n.e[(Y\ F)/f]$.

For mutual recursive functions, the encoding is a bit more complicated, but still possible.

Data

- The Haskell-program uses data types and selectors
- It suffices to represent booleans, tuples, lists of arbitrary length and natural numbers to encode the program
- We sketch the so-called Church encoding of numbers, booleans, pairs and lists.

Booleans

Encoding:

$$
\begin{array}{ll}
true & := \lambda x.\lambda y.x \\
false & := \lambda x.\lambda y.y
\end{array}
$$

$b\ s\ t$ for $b \in \{true, false\}$ behaves like `if` b `then` s `else` t.

Church Encoding of Numbers

Idea:

- Number i is represented by the i-fold function composition.
- $0 = f^0 = id$, $1 = f^1$, $2 = f^2$, $\ldots$

Encoding:

$$\begin{array}{ll} 0 & := \lambda f.\lambda x.x \\ i & := \lambda f.\lambda x.f^i\ x, \text{ if } i > 0 \end{array}$$

Church Encoding of Numbers (Cont'd)

Addition:
$plus = \lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x)$

Example:

$$\begin{array}{lll} plus\ 3\ 2 & = & \underbrace{(\lambda m.\lambda n.\lambda f.\lambda x.m\ f\ (n\ f\ x))}_{plus}\ \underbrace{(\lambda f.\lambda x.(f\ (f\ (f\ x))))}_{3}\ \underbrace{(\lambda f.\lambda x.(f\ (f\ x)))}_{2} \\ & \xrightarrow{C,\beta} & (\lambda n.\lambda f.\lambda x.(\lambda f.\lambda x.(f\ (f\ (f\ x))))\ f\ (n\ f\ x))\ (\lambda f.\lambda x.(f\ (f\ x))) \\ & \xrightarrow{C,\beta} & (\lambda n.\lambda f.\lambda x.(\lambda x.(f\ (f\ (f\ x))))\ (n\ f\ x))\ (\lambda f.\lambda x.(f\ (f\ x))) \\ & \xrightarrow{C,\beta} & (\lambda n.\lambda f.\lambda x.(f\ (f\ (f\ (n\ f\ x)))))\ (\lambda f.\lambda x.(f\ (f\ x))) \\ & \xrightarrow{C,\beta} & \lambda f.\lambda x.(f\ (f\ (f\ ((\lambda f.\lambda x.(f\ (f\ x)))\ f\ x)))) \\ & \xrightarrow{C,\beta,2} & \lambda f.\lambda x.(f\ (f\ (f\ (f\ (f\ x))))) = 5 \end{array}$$

Church Encoding of Numbers (Cont'd)

Successor:
$succ = \lambda n.\lambda f.\lambda x.f\ (n\ f\ x)$.

Predecessor (complicated and $pred\ 0 = 0$):
$pred = \lambda n.\lambda f.\lambda x.n\ (\lambda g.\lambda h.h\ (g\ f))\ (\lambda z.x)\ (\lambda u.u)$

Pairs

Encoding:
$pair := \lambda x.\lambda y.\lambda z.z\ x\ y$

The first two arguments are the arguments of the pair, the third one is for the selector.

Examples:

$$\begin{array}{ll} first & = \lambda p.p\ K \\ second & = \lambda p.p\ K2 \end{array}$$

Note that $first\ (pair\ s\ t) \sim_c s$ and $second\ (pair\ s\ t) \sim_c t$.

Lists

Non-empty lists can be encoded by pairs p:

- the first component of p is the element
- the second component of p is the tail of the list

With the empty list: additional pair $pair\ flag\ p$ where $flag$ is $true$ or $false$ and

- $pair\ true\ s$ means the empty list (independent of s)
- $(false, p)$ is a non-empty list.

Encoding:

$$\begin{array}{ll} nil & := pair\ true\ true \\ cons & := \lambda h.\lambda t.pair\ false\ (pair\ h\ t) \end{array}$$

Examples:

$$\begin{array}{ll} isNil & = first \\ head & = \lambda l.first\ (second\ l) \\ tail & = \lambda l.second\ (second\ l) \end{array}$$

Remarks

- Church encoding does not distinguish between data and functions
- Also different data is encoded in the same way (e.g. 0 and $false$)

Types?

- Haskell only allows well-typed programs
- the lambda calculus has no types
- This is not a restriction for Turing completeness, since the typed encoding can also be used as an untyped one

This concludes our sketch on Turing completeness

Discussion

- the lambda calculus is too small to really program in it
- also for a core language it is too difficult to express data and recursion
- equivalences in the lambda calculus do not necessarily hold in Haskell, since different data is mapped to the same lambda expressions
- Haskell has seq, the lambda calculus cannot simulate this
- In the next chapter: We extend the lambda calculus to a real core language of functional programming