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# Programming Language Foundations

## 02 Computability

Prof. Dr. David Sabel
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## Contents

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# INTUITIVE COMPUTABILITY

## Intuitive Computability

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- what is computable by a computer program compute, what not?
- do you have an intuition?
- we use the following definition:

**Definition (Intuitive Computability)**

A (partial) function $f : \mathbb{N}_0^k \to \mathbb{N}_0$ is computable iff there exists an algorithm (a program in a modern programming language) that computes $f$, i.e.
on input $(n_1, \ldots, n_k) \in \mathbb{N}_0^k$

- if $f(n_1, \ldots, n_k)$ is defined, then the program terminates after a finite number of steps and returns $f(n_1, \ldots, n_k)$ as result.
- if $f(n_1, \ldots, n_k)$ is undefined, then the program runs forever.

## Intuitive Computability: Examples

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**Computing the sum of two numbers**

Let $f : (\mathbb{N}_0 \times \mathbb{N}_0) \to \mathbb{N}_0$ be the function $f(x, y) = x + y$.

Is the function $f$ computable?

Yes, the algorithm with inputs $n_1, n_2 \in \mathbb{N}_0$ and program code:

$$\begin{array}{l} \mathit{result} := n_1 + n_2; \\ \texttt{return}\ \mathit{result}; \end{array}$$

computes $f$.



## Intuitive Computability: Examples

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**Always undefined function**

Let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be the partial function that is undefined for all inputs (often written as $f(x) = \perp$).

Is $f$ computable?

Yes, the algorithm with input $n \in \mathbb{N}_0$ and program code

$$\texttt{while true}\ \{\texttt{skip}\};$$

computes $f$.



## Intuitive Computability: Examples

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**Prefix of $\pi$**

Let $f(n) = \begin{cases} 1, & \text{if } n \text{ is a prefix of the digits of the decimal representation of } \pi \\ 0, & \text{otherwise} \end{cases}$

For example, $f(31) = 1 \quad f(314) = 1 \quad f(2) = 0 \quad f(315) = 0.$

Is $f$ computable?

Yes, the function $f$ is computable:

- There are algorithms that can compute the first $x$ digits of $\pi$
- Choose $x$ large enough to capture the digits of $n$
- Compare the digits with the digits of $n$ and return $1$ if all match, and $0$ otherwise



## Intuitive Computability: Examples

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**Substring of $\pi$**

Let $f(n) = \begin{cases} 1, & \text{if } n \text{ is a substring of the digits of the decimal representation of } \pi \\ 0, & \text{otherwise} \end{cases}$

Is $f$ computable?

- There is no known algorithm to check the condition
- If we would know, that $\pi$ contains every sequence of numbers (an open problem), then $f$ is trivially computable (always return 1)

## Intuitive Computability: Examples

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**Specific substring of $\pi$**

Let $f(n) = \begin{cases} 1, & \text{if the digits of the decimal representation of } \pi \\ & \text{contains the substring } 3^m \text{ for some number } m \geq n \\ 0, & \text{otherwise} \end{cases}$

Is $f$ computable?

The problem looks as hard as the previous one, but this is not the case.

1. If $\pi$ contains all strings $3^m$, then $f$ is the constant function 1, which is computable
2. If there is a bound $M$ such that $\pi$ contains $3^M$, but $\pi$ does not contain $3^x$ with $x > M$, then $f$ can be computed:
   Check if $n \leq M$ holds. If yes, return $1$, else return $0$.

One of both algorithms computes $f$, and thus $f$ is computable.

It is not relevant, that we do not know which one is the correct algorithm.



## Intuitive Computability: Examples

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**Function depending on open question**

Let $f$ be $f(n) = \begin{cases} 1, & \text{if } P = NP \\ 0, & \text{if } P \neq NP \end{cases}$

If $f$ computable?

Yes, because either $P = NP$ holds (then $f(n) = 1$ for all $n$), or $P \neq NP$ holds (then $f(n) = 0$ for all $n$).

Again, we do not know which algorithm is the right one, but we are sure that an algorithm that computes $f$ exists.



## Intuitive Computability: Examples

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**A lot functions**

Let $f^r$ be the function and $r$ be a real number

$$f^r(n) = \begin{cases} 1, & \text{if } n \text{ is prefix of the digits of the decimal representation of } r \\ 0, & \text{otherwise} \end{cases}$$

Are all functions $f^r$ computable?

No, the argument is:

- $f^{r_1} \neq f^{r_2}$ for $r_1 \neq r_2$
- $|\mathbb{R}|$ different algorithms are required
- the set of algorithms is countable
- the real numbers are not countable



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# TURING COMPUTABILTY

## Turing Machines: Informally

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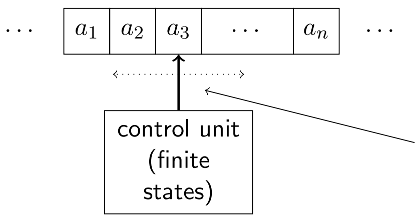


tape consisting of cells
(unbounded to the left and to the right)

read-/write-head
(may move to the left or to the right)

- introduced in 1936 by Alan Turing
- memory is represented by the infinite tape (divided into cells)
- in one step: TM reads the current cell, replaces the symbol, and may move the head by one cell



## Turing Machine, formally

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**Definition**

A Turing machine (TM) is a 7-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, \Box, F)$ where

- $Q$ is a finite, non-empty set of states,
- $\Sigma$ is a finite set of symbols, the input alphabet,
- $\Gamma \supset \Sigma$ is a finite set of symbols, the tape alphabet,
- $\delta$ is the state transition function where in the case of a deterministic Turing machine (DTM), $\delta : (Q \times \Gamma) \to (Q \times \Gamma \times \{L, R, N\})$, and in case of a non-deterministic Turing machine (NTM), $\delta : (Q \times \Gamma) \to \mathcal{P}(Q \times \Gamma \times \{L, R, N\})$,
- $q_0 \in Q$ is the start state,
- $\Box \in \Gamma \setminus \Sigma$ is the blank symbol,
- $F \subseteq Q$ is the set of final states.

For a deterministic Turing machine, an entry $\delta(q, a) = (q', b, x)$ means that in state $q$,



current cell will be $b$ and the read-/write-head moves into direction $x$ (which is left if $x = L$, right if $x = R$ and no move if $x = N$).
For a non-deterministic Turing machine the same holds if $(q', b, x) \in \delta(q, a)$, but it means, that the TM can do this, but it can also do some other state transition in $\delta(q, a)$. It chooses non-deterministically between the choices in $\delta(q, a)$.

## Configuration of TMs

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**Definition**

A configuration of a Turing machine is a word $wqw' \in \Gamma^* Q \Gamma^*$

A configuration $wqw'$ means

- the TM is in state $q$
- the tape content is $ww'$ and infinitely many blank symbols left and right from $ww'$
- the current head position is on the first symbol of $w'$.

Initially the TM is in state $q_0$ and the head is on the first symbol of the input word:

**Definition**

For input $w$, the start-configuration of a TM $M = (Q, \Sigma, \Gamma, \delta, q_0, \Box, F)$ is $q_0 w$.



## Transition Relation

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**Definition (Transition relation on configurations)**

For a TM $M = (Q, \Sigma, \Gamma, \delta, q_0, \Box, F)$, the relation $\vdash_M$ is defined as follows
(where $\delta(q, a) = (q', c, x)$ in case of an NTM means $(q', c, x) \in \delta(q, a)$):

| | | | |
|---|---|---|---|
| $wqw'$ | $\not\vdash_M$ | | if $q \in F$ (no transition for final states). |
| $b_1 \cdots b_m q a_1 \cdots a_n$ | $\vdash_M$ | $b_1 \cdots b_m q' c a_2 \cdots a_n$, | if $\delta(q, a_1) = (q', c, N)$, $m \geq 0, n \geq 1, q \notin F$ |
| $b_1 \cdots b_m q a_1 \cdots a_n$ | $\vdash_M$ | $b_1 \cdots b_{m-1} q' b_m c a_2 \cdots a_n$, | if $\delta(q, a_1) = (q', c, L)$, $m \geq 1, n \geq 1, q \notin F$ |
| $b_1 \cdots b_m q a_1 \cdots a_n$ | $\vdash_M$ | $b_1 \cdots b_m c q' a_2 \cdots a_n$, | if $\delta(q, a_1) = (q', c, R)$, $m \geq 0, n \geq 2, q \notin F$ |
| $b_1 \cdots b_m q a_1$ | $\vdash_M$ | $b_1 \cdots b_m c q' \Box$, | if $\delta(q, a_1) = (q', c, R)$ and $m \geq 0, q \notin F$ |
| $q a_1 \cdots a_n$ | $\vdash_M$ | $q' \Box c a_2 \cdots a_n$, | if $\delta(q, a_1) = (q', c, L)$ and $n \geq 1, q \notin F$ |

$\vdash_M^i$ is the $i$-fold application of $\vdash_M$
$\vdash_M^*$ the reflexive-transitive closure of $\vdash_M$
We omit the index $M$ in $\vdash_M$ and write $\vdash$ is $M$ is clear from the context.

## Example



DTM $M = (\{q_0, q_1, q_2, q_3\}, \{0,1\}, \{0,1,\Box\}, \delta, q_0, \Box, \{q_3\})$ with

$$\begin{array}{lll}
\delta(q_0,0) = (q_0,0,R) & \delta(q_0,1) = (q_0,1,R) & \delta(q_0,\Box) = (q_1,\Box,L) \\
\delta(q_1,0) = (q_2,1,L) & \delta(q_1,1) = (q_1,0,L) & \delta(q_1,\Box) = (q_3,1,N) \\
\delta(q_2,0) = (q_2,0,L) & \delta(q_2,1) = (q_2,1,L) & \delta(q_2,\Box) = (q_3,\Box,R) \\
\delta(q_3,0) = (q_3,0,N) & \delta(q_3,1) = (q_3,1,N) & \delta(q_3,\Box) = (q_3,\Box,N)
\end{array}$$

- interprets the input as binary number
- In state $q_0$ it moves the head to the right end and switches to $q_1$
- In $q_1$ it adds $1$ to the input, including a carryover
- If no more carryover occurs, it switches to $q_2$
- in $q_2$ it moves the head to the left end and switches to $q_3$
- it accepts in $q_3$



## Example execution



$$q_0 0011 \vdash 0q_0 011 \vdash 00q_0 11 \vdash 001q_0 1 \vdash 0011q_0\Box$$

$$\vdash 001q_1 1\Box \vdash 00q_1 10\Box \vdash 0q_1 000\Box \vdash q_2 0100\Box$$

$$\vdash q_2\Box 0100\Box \vdash \Box q_3 0100\Box$$



## Turing Computability



Let $bin(n)$ be the binary representation of number $n \in \mathbb{N}_0$.

**Definition**

Function $f : \mathbb{N}_0^k \to \mathbb{N}_0$ is Turing computable,
if there exists a DTM $M = (Q, \Sigma, \Gamma, \delta, q_0, \Box, F)$ such that for all $n_1, \ldots, n_k, m \in \mathbb{N}_0$:

$$f(n_1, \ldots, n_k) = m$$
iff
$$q_0\, bin(n_1)\#\ldots\#bin(n_k) \vdash^* \Box\ldots\Box q_f\, bin(m)\Box\ldots\Box \text{ with } q_f \in F.$$

Function $f : \Sigma^* \to \Sigma^*$ is Turing computable,
if there exists a DTM $M = (Q, \Sigma, \Gamma, \delta, q_0, \Box, F)$ such that for all $u, v \in \Sigma^*$:

$$f(u) = v \text{ iff } q_0 u \vdash^* \Box\ldots\Box q_f v\Box\ldots\Box \text{ with } q_f \in F.$$

If $f(n_1, \ldots, n_k)$ is undefined, we assume that the TM loops.



## Examples



- The successor function $f(x) = x + 1$ is Turing computable.

  We defined the corresponding TM in the last example.
- The identity $f(x) = x$ is Turing computable:

  DTM $M = (\{q_0\}, \{0,1,\#\}, \{0,1,\#\Box\}, \delta, q_0, \Box, \{q_0\})$ with $\delta(q_0, a) = (q_0, a, N)$ for all $a \in \{0,1,\#,\Box\}$, we have $q_0\, bin(n) \vdash^* q_0\, bin(n)$ for all $n \in \mathbb{N}_0$.
- The function $f(x) = \perp$ which is undefined for every input is Turing computable:

  DTM $M = (\{q_0\}, \{0,1,\#\}, \{0,1,\#,\Box\}, \delta, q_0, \Box, \emptyset)$ with $\delta(q_0, a) = (q_0, a, N)$ loops for every input and never reaches a final state.

## Not Turing Computable Functions

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- Turing machines and words can be encoded as numbers (called Gödel numbers)
- Let $f$ be a function that gets a number $n$ and
  - is undefined if $n$ is not a valid encoding of a TM $M$ and a word $w$
  - is 1, if the TM $M$ holds on input $w$
  - is 0, otherwise
- Let $f'$ be a function that gets a number $n$ and
  - is undefined if $n$ is not a valid encoding of a TM $M$ and a word $w$
  - is 1, if the TM $M$ holds on input $w$
  - is undefined, otherwise
- Function $f$ is **not Turing computable**, because the TM that computes $f$ has to solve the halting problem for Turing machines which is undecidable.
- Function $f'$ is **Turing computable**, because $f'$ can be computed by a Turing machine, by simulating $M$ on input $w$.



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## CHURCH-TURING-THESIS

## Computability

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In the 1930s also other notions of computability were invented, e.g.:

- Kurt Gödel and Jacques Herband: General recursive functions
- Alonzo Church and Stephen Kleene: $\lambda$-definable functions

Remarkable result:

All of the formalisms were shown to be equivalent, i.e. they define the same class of functions.



## Church-Turing Thesis

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**Church-Turing Thesis**

The class of Turing computable functions is identical to the class of intuitively computable functions.

- Thesis cannot be proved, since there is no formal definition of "intuitively computable".

## Turing Completeness

**Definition (Turing completeness**

A formalism (a programming language, an instruction set of a computer, a rewrite system etc.) is called Turing complete iff it can simulate a Turing machine.

Turing completeness means that every Turing computable function can also be computed by the formalism.

- several formalisms were shown to be Turing complete and thus they can be replaced by Turing computability in the Church-Turing thesis – since they all compute the same class of functions.
- Among them are all modern programming languages, the lambda-definable functions, the general recursive functions, WHILE-programs, GOTO-programs, the RAM-model, etc.
- You may convince yourself that your favourite programming language is Turing complete by programming a simulation of Turing machines.



## Conclusion

- We recalled Turing machines and Turing computability
- Several other formalisms are Turing-complete
- Church-Turing-Thesis: all these formalisms match the class of intuitively computable functions
- For considering foundational models of programming languages, we have several choices as long as the model is Turing-complete



## APPENDIX

- Gödel-numbering of Turing machines
- Undecidability of the halting problem

## Gödel-Numbering of Turing Machines

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, \Box, F)$ be a DTM with $\Sigma = \{0, 1\}$ and

- $\Gamma = \{a_0, \ldots, a_k\}$ where $a_0 = \Box$, $a_1 = \#$, $a_2 = 0$, $a_3 = 1$
- $Q = \{q_0, \ldots, q_n\}$
- $F = \{q_n\}$

For $\delta(q_r, a_s) = (q_t, a_u, D)$ generate a word over $\{0, 1, \#\}$:

$$w_{r,s,t,u,D} = \#\#bin(r)\#bin(s)\#bin(t)\#bin(u)\#bin(val(D))$$

with $val(L) = 0, val(R) = 1$, and $val(N) = 2$

For $M$ we generate $w_M$:

- Concatenate all words $w_{r,s,t,u,D}$ for $r \in \{0, \ldots, n\}, s \in \{0, \ldots, k\}$ and $t, u, D$ given by $\delta(q_r, a_s) = (q_t, a_u, D)$
- Apply the following encoding to each symbol $\{0 \mapsto 00, 1 \mapsto 01, \# \mapsto 11\}$

## Gödel-Numbering of Turing Machines (Cont'd)

- Not every word over $\{0,1\}$ is an encoding of a Turing machine (i.e. there exists $w$ such that $w \neq w_M$ for all TMs $M$)
- To fix this: Let $\widehat{M}$ be a fixed (but arbitrary) Turing machine.
- For $w \in \{0,1\}^*$ let $M_w$ be:

$$M_w := \begin{cases} M, & \text{if } w = w_M \\ \widehat{M}, & \text{otherwise} \end{cases}$$



## Undecidability of the Halting-Problem

The halting problem is $H := \{w_M\#w \mid \text{TM } M \text{ halts on input } w\}$

- Assumption: $H$ is decidable, i.e. there exists a TM $M_H$ that terminates for any input $w_M\#w$ with output
  - $1$ (=Yes) if $M$ halts on input $w$
  - $0$ (=No) if $M$ does not halt on input $w$

Using $M_H$ and the Gödel-numbering we can fill an (infinite) table, with entries Yes or No, depending on whether or not $M_i$ halts on input $w_{M_j}$

| | $w_{M_1}$ | $w_{M_2}$ | $w_{M_3}$ | $\dots$ |
|---|---|---|---|---|
| $M_1$ | $Yes$ | $No$ | $\dots$ | |
| $M_2$ | $No$ | $No$ | $\dots$ | |
| $M_3$ | $Yes$ | $No$ | $\dots$ | |
| $\dots$ | $\dots$ | $\dots$ | $\dots$ | |



We construct a TM $M_K$:

- On input $w$, it checks whether $M_w$ holds on $w$ by using $M_H$.
- If yes, then $M_K$ loops
- If no, then $M_K$ stops successfully.

By construction:

$$M_K \text{ halts on } w_{M_j} \text{ iff } M_j \text{ does not halt on } w_{M_j}$$

Since all TMs are in the table: there is a $j$ such that $M_j = M_K$.

$$M_j \text{ halts on input } w_{M_j} \text{ iff } M_j \text{ does not halt on input } w_{M_j}$$

This is a contradiction!
Our assumption was wrong: The halting problem is not decidable.