Improvements in a Call-by-Need Functional Core Language: Common Subexpression Elimination and Resource Preserving Translations

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Abstract

An improvement is a correct program transformation that optimizes the program, where the criterion is that the number of computation steps until a value is obtained is not strictly increased in any context. This paper investigates improvements in both – an untyped and a polymorphically typed variant – of a call-by-need lambda calculus with letrec, case, constructors and seq. Besides showing that several local transformations are optimizations, a main result of this paper is a proof that common subexpression elimination is correct and an improvement, which proves a conjecture and thus closes a gap in the improvement theory of Moran and Sands. The improvement relation used in this paper is generic in which essential computation steps are counted and thus the obtained results apply for several notions of improvement. Besides the small-step operational semantics, also an abstract machine semantics is considered for counting computation steps. We show for several length measures that the call-by-need calculus of Moran and Sands and our calculus are equivalent.

Keywords: semantics, lambda calculus, functional programming, lazy evaluation, improvement

1. Introduction

Motivation and State of the Art. Lazy functional programming languages like Haskell [9] support declarative programming, since they provide a high level of abstraction and allow a definition of the intended results without specifying the exact sequence of operations (see for instance, [6] for further motivation of functional programming). While there does not exist an official formal semantics of Haskell, it is often loosely identified with an extended lazy lambda calculus with call-by-name evaluation. However, all real implementations of Haskell use call-by-need evaluation – i.e. lazy evaluation extended with sharing to avoid duplicated evaluation of subexpressions.

For reasoning about program semantics, it does not matter whether a call-by-name or call-by-need semantics is used, since both induce the same (equational) semantics. However, for reasoning about the resource consumption, the call-by-name evaluation does not match the real implementations and thus a call-by-need has to be used. Thus, call-by-need program calculi provide a good model of both the correctness of the programs as well as the amount of required work for executing programs. Analyzing these calculi and providing tools for proving transformations correct and/or to be optimizations is cumbersome, since sharing complicates reasoning, but it is worth the effort.

There are several works on analyzing and proving correctness of program transformations (e.g. [11, 7, 24]). However, there seems to be much less research on whether the (correct) program transformations are also optimizations – i.e. while preserving the meaning of the programs they also do not increase the runtime or the space behavior of the programs. Having such results is for instance useful in automated tools for program transformation like Hermit [25] and in general for optimizing compilers.

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A theory of optimizations or improvements in extended lambda calculi was developed in [10] for an untyped higher-order language with call-by-need evaluation, and for a call-by-value variant in [16]. In [10] the resource model counts the steps of an abstract machine for call-by-need evaluation which is a variant of Sestoft's abstract machine [26]. The work of Moran and Sands [10] provides a foundation for program improvements which leads to several results exhibiting program transformations that are improvements and also provides techniques for showing program transformations being improvements. Moran and Sands also remark in [10] that the reductions used in any context (a form of partial evaluation) are improvements, but the efficiency gain has a limit: it is at most polynomial. A detailed analysis on this topic can be found in the work of Gustavsson and Sands [4] for a call-by-value lambda calculus. Clearly, other program transformations (which are not calculus reductions) have a higher potential to improve efficiency. One such rule is common subexpression elimination which identifies equal subexpressions of the program and replaces them by references to a single copy of the subexpression. As a small (artificial) example with an exponential speedup, consider the (inefficient) definition of the following tree-recursive function `treeRec` in Haskell:

```
treeRec x = if x == 0 then 1 else treeRec (x-1) + treeRec (x-1)
```

Evaluation of `treeRec` n requires $O(2^n)$ reduction steps. Applying common subexpression elimination for the two occurrences of `treeRec (x-1)` results in the definition

```
treeRec x = if x == 0 then 1 else let y = treeRec (x-1) in y + y
```

for which the evaluation of `treeRec` n only requires $O(n)$ reduction steps.

Common subexpression elimination is treated in [10], but not proved to be an improvement (but it is conjectured).

Recently, Hackett and Hutton [5] rediscovered the improvement theory of [10] to argue that optimizations are indeed improvements, with a particular focus on worker/wrapper transformations (see for instance [2] for more examples). The work of [5] is based on the call-by-need abstract machine of [10] with a slightly modified measure for the improvement relation.

Goals and Results. The so-called LR-calculus (an extended lambda calculus with letrec) – introduced in [24] – is an extended higher-order lambda calculus which models the core language of Haskell. It extends the lambda calculus with `letrec`-expressions (to express recursive and shared-bindings), data constructors and `case`-expressions (which act as selectors), and Haskell's `seq`-operator. The operational semantics of the LR-calculus is a small-step call-by-need evaluation. The goal of this paper is to develop an improvement theory for the LR-calculus.

Differences to the work of Moran and Sands are (i) that the LR-calculus uses a small-step operational semantics expressed by rewriting rules and a strategy, (ii) that LR does not restrict the syntax of arguments (of applications) to be variables (i.e. in LR arbitrary expressions are allowed as arguments), and (iii) that it includes the `seq`-operator for strict evaluation of expressions which is indispensable to model the semantics of Haskell (see e.g. [7, 17]).

We use previous results and techniques for the LR-calculus to establish new improvement laws, in particular we show that common subexpression elimination is an improvement. Here we can build upon a detailed analysis of reduction lengths (performed in [24] in the context of a strictness analysis); the method of using diagrams to compute and join overlaps between reductions and transformations which we developed and applied in several works [8, 24, 14, 15] to show correctness of program transformations; and correctness of inlining (or common subexpression elimination) via infinite expressions (to unfold and remove the `letrec`-expressions) established in [23]. We prefer analyzing reductions in LR, due to the success of the diagram method in LR. For example, the letrec calculus mentioned in [1] is related, but does not model case, constructors nor seq, and its deref rule complicates diagram proofs.

Since our improvement relation is different from [10] (it uses a different measure and operational semantics), we compare our measures with those in [10, 5]. The result is that our improvement theory can be transferred to the abstract machine of [10] using our measure, and that our calculus and Moran and Sand's calculus together with their measures are equivalent w.r.t. resources.

Analyzing untyped calculi covers a large amount of program transformations, however, sometimes typing arguments are required for showing that interesting program transformations are improvements (see e.g. [5]). Due to cyclic bindings, using monomorphic typing and monomorphising a polymorphic calculus is insufficient. Hence we adapt ideas from system-F polymorphism [3, 12], in particular from an intermediate language in a Haskell compiler [9, 27], and extend our improvement theory for the calculus LRP – a polymorphically typed variant of LR with `let`-polymorphism.

Since the type erasure of reduction sequences in LRP exactly leads to the untyped reduction sequences in LR, the connection between the untyped and the typed calculi is quite close and indeed it is rather simple to transfer the results and the techniques from LR into LRP. As a further application of our improvement theory for the typed setting, we consider a transformation (called (caseId)), which is type-dependent, i.e. the transformation is only correct in LRP, but incorrect in the untyped calculus LR. We show that our methods are applicable for (caseId) and show that the transformation is indeed an improvement.

Outline. In Sect. 2 we recall the untyped calculus LR, and in Sect. 3 we introduce improvement for LR and prove a context lemma. In Sect. 4 we show that common subexpression elimination is an improvement. In Sect. 5 we compare our length measure with the measures used by Moran and Sands' improvement theory. In Sect. 6 we consider the polymorphically typed variant of LR. We conclude in Sect. 7.

This paper is an extended version of [20] with the following additions and differences: The reduction measures `rln` (for the calculus) and `mln` for the abstract machine are parametrized, and thus all claims are generalized and re-proved. This extension to different length measures provides finer grained information on the complexity of functions. Furthermore, the analysis and comparison on (*lll*), (*gc*), (*cpax*)-normalized expressions in Sect. 5.4 is new. Also discussion on complexity of functions in Sect. 5.5 w.r.t. several measures is added compared to [20]. Finally, improvement results for the polymorphically typed calculus LRP in Section 6 are new.

2. The Call-by-Need Lambda Calculus LR

We recall the calculus LR [24], which is an untyped call-by-need lambda calculus that extends the lambda calculus by recursive `letrec`, data constructors, case-expressions, and the `seq`-operator. We recall results from previous investigations: from [24] we reuse a counting theorem for reduction lengths and correctness of several program transformations. From [23] we reuse correctness of copying arbitrary expressions.

We employ the syntax of the calculus LR [24]. Let *Var* be a countable infinite set of variables. We assume that there is a fixed set of type constructors $\mathcal{K}$, where every type constructor $K \in \mathcal{K}$ has an arity $ar(K) \geq 0$, and there is a finite, non-empty set $D_K = \{c_{K,1}, \ldots, c_{K,|D_K|}\}$ of data constructors. Every data constructor has an arity $ar(c_{K,i}) \geq 0$.

Example 2.1. *$\mathcal{K}$ may include the 0-ary type constructor* `Bool` *and the 1-ary type constructor* `List` *with* $D_{\texttt{Bool}} = \{\texttt{True}, \texttt{False}\}$, $D_{\texttt{List}} = \{\texttt{Nil}, \texttt{Cons}\}$, $ar(\texttt{True}) = ar(\texttt{False}) = ar(\texttt{Nil}) = 0$, *and* $ar(\texttt{Cons}) = 2$.

The syntax of expressions $r, s, t \in Expr$ of LR is defined in Fig. 1. We write $FV(s)$ for the set of free variables of an expression s. Besides *variables* x, *abstractions* $\lambda x.s$, and *applications* $(s\ t)$ the syntax of LR comprises the following constructs: *Constructor applications* $(c_{K,i}\ s_1 \ldots s_{ar(c_{K,i})})$ always occur fully saturated. In our meta-notation we sometimes omit the index of the constructor or use vector notation and thus write for instance $(c\ \overrightarrow{s})$ instead of $(c_{K,i}\ s_1 \ldots s_{ar(c_{K,i})})$. In a `letrec`-*expression* $\texttt{letrec}\ x_1 = s_1, \ldots, x_n = s_n\ \texttt{in}\ t$ all variables $x_1, \ldots, x_n$ must be pairwise distinct, the scope of x_i is all s_i and t. The bindings $x_1 = s_1, \ldots, x_n = s_n$ are called the `letrec`-*environment* and t is called the `in`-*expression*. Sometimes the environment or parts of it are abbreviated by *Env* (for instance, we write `letrec` *Env* `in` s or `letrec` Env_1, Env_2 `in` s). For a `letrec`-environment $Env = \{x_1 = t_1, \ldots, x_n = t_n\}$, we define $LV(Env) := \{x_1, \ldots, x_n\}$ and sometimes write $\{x_i = t_i\}_{i=1}^n$ as abbreviation for such an environment. For a chain of variable-to-variable bindings $x_j = x_{j-1}, x_{j+1} = x_j, \ldots, x_m = x_{m-1}$ we use the abbreviation $\{x_i = x_{i-1}\}_{i=j}^m$. A `seq`-*expression* $(\texttt{seq}\ s\ t)$ can be used for strict evaluation of expressions, since the expression s must be successfully evaluated before t is evaluated. For every $K \in \mathcal{K}$ there

$$r, s, t \in \mathit{Expr} := x \mid \lambda x.s \mid (s\ t) \mid (c_{K,i}\ s_1\ \ldots\ s_{ar(c_{K,i})}) \mid (\texttt{letrec}\ x_1 = s_1, \ldots, x_n = s_n\ \texttt{in}\ t) \mid (\texttt{seq}\ s\ t)$$
$$\mid (\texttt{case}_K\ s\ (c_{K,1}\ x_1 \ldots x_{ar(c_{K,1})}\ \texttt{->}\ t_1) \ldots (c_{K,|D_K|}\ x_1 \ldots x_{ar(c_{K,|D_K|})}\ \texttt{->}\ t_{|D_K|}))$$

Figure 1: Expressions of the language LR where $x, x_i \in \mathit{Var}$ are term variables

$$\begin{array}{ll}
(s\ t)^{\mathsf{sub}\vee\mathsf{top}} & \to (s^{\mathsf{sub}}\ t)^{\mathsf{vis}} \\
(\texttt{letrec}\ \mathit{Env}\ \texttt{in}\ s)^{\mathsf{top}} & \to (\texttt{letrec}\ \mathit{Env}\ \texttt{in}\ s^{\mathsf{sub}})^{\mathsf{vis}} \\
(\texttt{letrec}\ x = s, \mathit{Env}\ \texttt{in}\ C[x^{\mathsf{sub}}]) & \to (\texttt{letrec}\ x = s^{\mathsf{sub}}, \mathit{Env}\ \texttt{in}\ C[x^{\mathsf{vis}}]) \\
(\texttt{letrec}\ x = s, y = C[x^{\mathsf{sub}}], \mathit{Env}\ \texttt{in}\ t) & \to (\texttt{letrec}\ x = s^{\mathsf{sub}}, y = C[x^{\mathsf{vis}}], \mathit{Env}\ \texttt{in}\ t),\ \text{if } C \neq [\cdot] \\
(\texttt{letrec}\ x = s, y = x^{\mathsf{sub}}, \mathit{Env}\ \texttt{in}\ t) & \to (\texttt{letrec}\ x = s^{\mathsf{sub}}, y = x^{\mathsf{nontarg}}, \mathit{Env}\ \texttt{in}\ t) \\
(\texttt{seq}\ s\ t)^{\mathsf{sub}\vee\mathsf{top}} & \to (\texttt{seq}\ s^{\mathsf{sub}}\ t)^{\mathsf{vis}} \\
(\texttt{case}\ s\ \texttt{of}\ \mathit{alts})^{\mathsf{sub}\vee\mathsf{top}} & \to (\texttt{case}\ s^{\mathsf{sub}}\ \texttt{of}\ \mathit{alts})^{\mathsf{vis}} \\
\texttt{letrec}\ x = s^{\mathsf{vis}\vee\mathsf{nontarg}}, y = C[x^{\mathsf{sub}}], \mathit{Env}\ \texttt{in}\ t & \to \texttt{Fail} \\
(\texttt{letrec}\ x = C[x^{\mathsf{sub}}], \mathit{Env}\ \texttt{in}\ s) & \to \texttt{Fail}
\end{array}$$

Figure 2: Computing reduction positions using labels (see Definition 2.4), where $a \vee b$ means label a or label b. The algorithm does not overwrite non-displayed labels.

(lbeta) $C[((\lambda x.s)^{\mathsf{sub}}\ r)] \to C[(\texttt{letrec}\ x = r\ \texttt{in}\ s)]$

(cp-in) $(\texttt{letrec}\ x_1 = (\lambda x.s)^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^m, \mathit{Env}\ \ \texttt{in}\ C[x_m^{\mathsf{vis}}])$
$\to (\texttt{letrec}\ x_1 = (\lambda x.s), \{x_i = x_{i-1}\}_{i=2}^m, \mathit{Env}\ \ \texttt{in}\ C[\lambda x.s])$

(cp-e) $(\texttt{letrec}\ x_1 = (\lambda x.s)^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^m, \mathit{Env}, y = C[x_m^{\mathsf{vis}}]\ \texttt{in}\ r)$
$\to (\texttt{letrec}\ x_1 = (\lambda x.s), \{x_i = x_{i-1}\}_{i=2}^m, \mathit{Env}, y = C[\lambda x.s]\ \texttt{in}\ r)$

(llet-in) $(\texttt{letrec}\ \mathit{Env}_1\ \texttt{in}\ (\texttt{letrec}\ \mathit{Env}_2\ \texttt{in}\ r)^{\mathsf{sub}}) \to (\texttt{letrec}\ \mathit{Env}_1, \mathit{Env}_2\ \texttt{in}\ r)$

(llet-e) $(\texttt{letrec}\ \mathit{Env}_1, x = (\texttt{letrec}\ \mathit{Env}_2\ \texttt{in}\ s_x)^{\mathsf{sub}}\ \texttt{in}\ r) \to (\texttt{letrec}\ \mathit{Env}_1, \mathit{Env}_2, x = s_x\ \texttt{in}\ r)$

(lapp) $C[((\texttt{letrec}\ \mathit{Env}\ \texttt{in}\ t)^{\mathsf{sub}}\ s)] \to C[(\texttt{letrec}\ \mathit{Env}\ \texttt{in}\ (t\ s))]$

(lcase) $C[\texttt{case}_K\ (\texttt{letrec}\ \mathit{Env}\ \texttt{in}\ t)^{\mathsf{sub}}\ \mathit{alts}] \to C[(\texttt{letrec}\ \mathit{Env}\ \texttt{in}\ \texttt{case}_K\ t\ \mathit{alts})]$

(lseq) $C[\texttt{seq}\ (\texttt{letrec}\ \mathit{Env}\ \texttt{in}\ s)^{\mathsf{sub}}\ t] \to C[(\texttt{letrec}\ \mathit{Env}\ \texttt{in}\ \texttt{seq}\ s\ t)]$

(seq-c) $C[\texttt{seq}\ v^{\mathsf{sub}}\ t] \to C[t],$ if v is a value

(seq-in) $\texttt{letrec}\ x_1 = (c\ \overrightarrow{t})^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^m, \mathit{Env}\ \texttt{in}\ C[\texttt{seq}\ x_m^{\mathsf{vis}}\ t]$
$\to \texttt{letrec}\ x_1 = (c\ \overrightarrow{t}), \{x_i = x_{i-1}\}_{i=2}^m, \mathit{Env}\ \texttt{in}\ C[t]$

(seq-e) $\texttt{letrec}\ x_1 = (c\ \overrightarrow{t})^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^m, \mathit{Env}, y = C[\texttt{seq}\ x_m^{\mathsf{vis}}\ t]\ \texttt{in}\ r$
$\to \texttt{letrec}\ x_1 = (c\ \overrightarrow{t}), \{x_i = x_{i-1}\}_{i=2}^m, \mathit{Env}, y = C[t]\ \texttt{in}\ r$

(case-c) $C[\texttt{case}_K\ (c\ \overrightarrow{t})^{\mathsf{sub}} \ldots (c\ \overrightarrow{y}\ \texttt{->}\ t) \ldots] \to C[\texttt{letrec}\ \{y_i = t_i\}_{i=1}^{ar(c)}\ \texttt{in}\ t],$ if $ar(c) \geq 1$

(case-c) $C[\texttt{case}_K\ c^{\mathsf{sub}}\ \ \ldots\ \ (c\ \texttt{->}\ t) \ldots] \to C[t],$ if $ar(c) = 0$

(case-in) $\texttt{letrec}\ x_1 = (c\ \overrightarrow{t})^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^m, \mathit{Env}\ \texttt{in}\ C[\texttt{case}_K\ x_m^{\mathsf{vis}} \ldots (c\ \overrightarrow{z}\ \texttt{->}\ t) \ldots]$
$\to \texttt{letrec}\ x_1 = (c\ \overrightarrow{y}), \{y_i = t_i\}_{i=1}^{ar(c)}, \{x_i = x_{i-1}\}_{i=2}^m, \mathit{Env}\ \texttt{in}\ C[\texttt{letrec}\ \{z_i = y_i\}_{i=1}^{ar(c)}\ \texttt{in}\ t],$
where $ar(c) \geq 1$ and y_i are fresh variables

(case-in) $\texttt{letrec}\ x_1 = c^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^m, \mathit{Env}\ \texttt{in}\ C[\texttt{case}_K\ x_m^{\mathsf{vis}} \ldots (c\ \texttt{->}\ t) \ldots]$
$\to \texttt{letrec}\ x_1 = c, \{x_i = x_{i-1}\}_{i=2}^m, \mathit{Env}\ \texttt{in}\ \ C[t],$ if $ar(c) = 0$

(case-e) $\texttt{letrec}\ x_1 = (c\ \overrightarrow{t})^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^m, u = C[\texttt{case}_K\ x_m^{\mathsf{vis}} \ldots (c\ \overrightarrow{z}\ \texttt{->}\ r_1) \ldots], \mathit{Env}\ \texttt{in}\ r_2$
$\to \texttt{letrec}\ x_1 = (c\ \overrightarrow{y}), \{y_i = t_i\}_{i=1}^{ar(c)}, \{x_i = x_{i-1}\}_{i=2}^m, u = C[\texttt{letrec}\ \{z_i = y_i\}_{i=1}^{ar(c)}\ \texttt{in}\ r_1], \mathit{Env}\ \texttt{in}\ r_2$
where $ar(c) \geq 1$ and y_i are fresh variables

(case-e) $\texttt{letrec}\ x_1 = c^{\mathsf{sub}}, \{x_i = x_{i-1}\}_{i=2}^m, u = C[\texttt{case}_K\ x_m^{\mathsf{vis}} \ldots (c\ \texttt{->}\ r_1) \ldots], \mathit{Env}\ \texttt{in}\ r_2$
$\to \texttt{letrec}\ x_1 = c, \{x_i = x_{i-1}\}_{i=2}^m \ldots, u = C[r_1], \mathit{Env}\ \texttt{in}\ r_2,$ if $ar(c) = 0$

Figure 3: Reduction rules

is a `case`-*expression* ($\texttt{case}_K\ s\ (c_{K,1}\ x_1 \ldots x_{ar(c_{K,1})}\ \texttt{->}\ t_1)\ \ldots\ (c_{K,|D_K|}\ x_1 \ldots x_{ar(c_{K,|D_K|})}\ \texttt{->}\ t_{|D_K|})$) with exactly one `case`-*alternative* $((c_{K,i}\ x_1 \ldots x_{ar(c_{K,i})})\ \texttt{->}\ t_i)$ for every data constructor $c_{K,i} \in D_K$. The variables $x_1, \ldots, x_{ar(c_{K,i})}$ in the `case`-*pattern* $((c_{K,i}\ x_1 \ldots x_{ar(c_{K,i})})\ \texttt{->}\ t_i)$ must be pairwise distinct and the scope of the variables $x_1, \ldots, x_{ar(c_{K,i})}$ is the expression t_i. We sometimes use the meta-symbol *alts* to abbreviate the `case`-alternatives and thus write ($\texttt{case}_K\ s\ alts$).

Example 2.2. *The* `case`*-expression* $\texttt{case}_{\texttt{Bool}}\ s_1\ (\texttt{True}\ \texttt{->}\ s_2)\ (\texttt{False}\ \texttt{->}\ s_3)$ *corresponds to the conditional* `if` s_1 `then` s_2 `else` s_3*. The function map which maps a function to all elements of a list can be defined as* $\texttt{letrec}\ map = \lambda f, xs.\texttt{case}_{\texttt{List}}\ xs\ (\texttt{Nil}\ \texttt{->}\texttt{Nil})\ (\texttt{Cons}\ y\ ys\ \texttt{->}\texttt{Cons}\ (f\ y)\ (map\ f\ ys))\ \texttt{in}\ map$.

Definition 2.3. *A* context C *is an expression with a hole (denoted by* $[\cdot]$*) at expression position.* Surface contexts S *are contexts where the hole is not in an abstraction,* top contexts T *are surface contexts where the hole is not in an alternative of a* `case`*, and* weak top contexts *are top contexts where the hole is not in a* `letrec`*-expression. With* $C[s]$ *we denote the substitution of the hole in the context* C *by expression* s*. A* multicontext M *is an expression with zero or more (different) holes at expression positions.*

We assume the convention that bound variables are only introduced once in expressions, and that these are different from free variables, which can always be achieved by renaming bound variables. We assume that reduction rules and transformations are followed by a renaming to fulfill this distinct variable convention. In our meta-notations we assume that variable names that appear twice are the same object, if not mentioned otherwise. For instance, the notation ($\texttt{letrec}\ x = s, Env\ \texttt{in}\ C[x^{\mathsf{sub}}]$) assumes that C does not capture x in its hole.

2.1. Normal Order Reduction

A *value* in LR is an abstraction $\lambda x.s$ or a constructor application $(c\ \overrightarrow{s})$. The reduction rules of LR are defined in Fig. 3, where the role of the labels $\mathsf{sub}, \mathsf{top}, \mathsf{vis}, \mathsf{nontarg}$ will be explained below in Definition 2.4. The rule (lbeta) is the sharing variant of classical β-reduction. The rules (cp-in) and (cp-e) copy abstractions. The rules (llet-in) and (llet-e) join two `letrec`-environments. The rules (lapp), (lcase), and (lseq) float-out a `letrec` from the first argument of an application, a `case`-, or a `seq`-expression. The rules (seq-c), (seq-in), and (seq-e) evaluate a `seq`-expression, provided that the first argument is a value (or a variable that is bound (via indirections) to a constructor application). The rules (case-c), (case-in), and (case-e) evaluate a `case`-expression provided that the first argument is (or is a variable which is bound to) a constructor application of the right type.

The normal order reduction strategy of the calculus LR is a call-by-need strategy. It applies the reduction rules at specific positions which will be defined by an algorithm to find the position of a redex[2].

Definition 2.4 (Labeling Algorithm)**.** *The labeling algorithm detects the position to which a reduction rule will be applied according to normal order. It uses the labels* top,sub,vis,$\mathsf{nontarg}$ *where* top *means reduction of the top term,* sub *means reduction of a subterm,* vis *marks already visited subexpressions, and* $\mathsf{nontarg}$ *marks visited variables that are not target of a (cp)-reduction. For a term* s *the labeling algorithm starts with* s^{top}*, where no other subexpression in* s *is labeled and proceeds by applying the rules given in Fig. 2 exhaustively.*

Note that the labeling algorithm does not descend into sub-labeled `letrec`-expressions. If the labeling algorithm does not fail, then a potential normal order redex is found, which can only be a superterm of the sub-marked subexpression. However, it is possible that there is no normal order reduction, if the evaluation is already finished, or no rule is applicable.

[2]This is equivalent to a syntactic definition of reduction contexts (using a context free grammar) which can be found for the calculus LR in [24]. In this paper we decided to use the definition based on the labeling algorithm, since the syntactic description using reduction contexts is quite complex, for instance, for rule (no,cp-e) the shape of the reduced expression is in general of the form $\texttt{letrec}\ x_1 = \lambda x.s, x_2 = x_1, \ldots, x_m = x_m - 1, y_1 = A_1[x_m], y_2 = A_2[y_1], \ldots y_n - 1 = A_{n-1}[y_{n-2}], Env\ \texttt{in}\ A_n[y_n]$ where A_i are reduction contexts without `letrec`, defined by the grammar $A ::= [.]|(A\ s)|\texttt{case}_T\ A\ alts)\ |\ \texttt{seq}\ A\ s$ and context A_1 is non-empty.

(gc1) $\texttt{letrec}\ \{x_i = s_i\}_{i=1}^{n}, Env\ \texttt{in}\ t \to \texttt{letrec}\ Env\ \texttt{in}\ t,$ if for all $i : x_i \notin FV(t, Env)$

(gc2) $\texttt{letrec}\ \{x_i = s_i\}_{i=1}^{n}\ \texttt{in}\ t \to t,$ if for all $i : x_i \notin FV(t)$

(cpx-in) $\texttt{letrec}\ x = y, Env\ \texttt{in}\ C[x] \to \texttt{letrec}\ x = y, Env\ \texttt{in}\ C[y],$ if $x \neq y \in Var$

(cpx-e) $\texttt{letrec}\ x = y, z = C[x], Env\ \texttt{in}\ t \to \texttt{letrec}\ x = y, z = C[y], Env\ \texttt{in}\ t,$ if $x \neq y \in Var$

(cpax) $\texttt{letrec}\ x = y, Env\ \texttt{in}\ s \to \texttt{letrec}\ x = y, Env[y/x]\ \texttt{in}\ s[y/x],$ if $x \neq y \in Var,\ y \in FV(s, Env)$

(cpcx-in) $\texttt{letrec}\ x = c\,\overrightarrow{t}, Env\ \texttt{in}\ C[x] \to \texttt{letrec}\ x = c\,\overrightarrow{y}, \{y_i = t_i\}_{i=1}^{ar(c)}, Env\ \texttt{in}\ C[c\,\overrightarrow{y}]$

(cpcx-e) $\texttt{letrec}\ x = c\,\overrightarrow{t}, z = C[x], Env\ \texttt{in}\ t \to \texttt{letrec}\ x = c\,\overrightarrow{y}, \{y_i = t_i\}_{i=1}^{ar(c)}, z = C[c\,\overrightarrow{y}], Env\ \texttt{in}\ t$

(abs) $\texttt{letrec}\ x = c\,\overrightarrow{t}, Env\ \texttt{in}\ s \to \texttt{letrec}\ x = c\,\overrightarrow{x}, \{x_i = t_i\}_{i=1}^{ar(c)}, Env\ \texttt{in}\ s,$ where $ar(c) \geq 1$

(abse) $(c\,\overrightarrow{t}) \to \texttt{letrec}\ \{x_i = t_i\}_{i=1}^{ar(c)}\ \texttt{in}\ c\,\overrightarrow{x},$ where $ar(c) \geq 1$

(xch) $\texttt{letrec}\ x = t, y = x, Env\ \texttt{in}\ r \to \texttt{letrec}\ y = t, x = y, Env\ \texttt{in}\ r$

(ucp1) $\texttt{letrec}\ Env, x = t\ \texttt{in}\ S[x] \to \texttt{letrec}\ Env\ \texttt{in}\ S[t]$

(ucp2) $\texttt{letrec}\ Env, x = t, y = S[x]\ \texttt{in}\ r \to \texttt{letrec}\ Env, y = S[t]\ \texttt{in}\ r$

(ucp3) $\texttt{letrec}\ x = t\ \texttt{in}\ S[x] \to S[t]$

where in the (ucp)-rules, x has at most one occurrence in $S[x]$ and no occurrence in Env, t, r; and S is a surface context

Figure 4: Extra transformation rules

Definition 2.5 (Normal Order Reduction of LR)**.** *Let t be an expression. Then a single* normal order reduction step $t \xrightarrow{no} t'$ *is defined by first applying the labeling algorithm to t, and if the labeling algorithm terminates successfully, then one of the rules in Fig. 3 has to be applied, if possible, where the labels* sub, vis *must match the labels in the expression t (t may have more labels), and t' is the result after erasing all labels.*

Example 2.6. *For* $(\texttt{letrec}\ w{=}\lambda x.x, y{=}w, z{=}(y\ y)\ \texttt{in}\ z)$*, the labeling algorithm successfully ends with* $(\texttt{letrec}\ w{=}(\lambda x.x)^{\mathsf{sub}}, y{=}w^{\mathsf{nontarg}}, z{=}(y^{\mathsf{vis}}\ y)^{\mathsf{vis}}\ \texttt{in}\ z^{\mathsf{vis}})^{\mathsf{vis}}$*. Only the rule (cp-e) matches the subexpressions and the labels, i.e.* $(\texttt{letrec}\ w = \lambda x.x, y = w, z = (y\ y)\ \texttt{in}\ z) \xrightarrow{no} (\texttt{letrec}\ w = \lambda x.x, y = w, z = ((\lambda x'.x')\ y)\ \texttt{in}\ z)$ *where the α-renaming is performed implicitly to fulfill the distinct variable convention. Note that the label* nontarg *at w prevents the normal order reduction to copy the* sub*-labeled abstraction into this position.*

For $\texttt{letrec}\ x = (y\ y), y = x\ \texttt{in}\ y$ *the labeling fails and thus the expression has no normal order reduction. For* $(\texttt{Cons True Nil})\ (\lambda x.x)$*, the labeling terminates with* $((\texttt{Cons True Nil})^{\mathsf{sub}}\ (\lambda x.x))^{\mathsf{vis}}$ *and also for* $\texttt{letrec}\ x = \texttt{True}\ \texttt{in}\ x$*, the labeling ends successful with* $(\texttt{letrec}\ x = \texttt{True}^{\mathsf{sub}}\ \texttt{in}\ x^{\mathsf{vis}})^{\mathsf{vis}}$*. However, for both expressions no normal order reduction is applicable (since the labels do not match the reduction rules). The former expression is irreducible, since it has a "dynamic type error". The latter expression is irreducible since it is successfully evaluated, i.e. it is a weak head normal form (see Definition 2.8).*

By a case analysis one can verify that normal order reduction is unique, i.e. for an expression t either no normal order reduction is possible, or there is a unique expression t' (up-to α-equivalence) s.t. $t \xrightarrow{no} t'$.

We sometimes attach more information to the reduction arrow, for instance $\xrightarrow{no,lbeta}$ denotes a normal order reduction using rule (lbeta). For a binary relation $\to$, we write $\xrightarrow{+}$ for the transitive closure, and $\xrightarrow{*}$ for the reflexive-transitive closure of $\to$. E.g., $\xrightarrow{no,*}$ denotes the reflexive-transitive closure of $\xrightarrow{no}$. We write $\xrightarrow{n}$ for exactly n $\to$-steps and we write $\xrightarrow{n \vee m}$ for either n or m steps. The notation $\xrightarrow{a \vee b}$ is also used for a and b being rule names, meaning the union of the rules a and b. For instance, $\xrightarrow{no,lbeta \vee no,lapp,0 \vee 1}$ means one or none normal order reduction step using rule (lbeta) or rule (lapp). For two binary relations $\to_1, \to_2$, we write $\to_1 . \to_2$ for their composition, that is $\to_1 . \to_2 := \{(s_1, s_3) \mid s_1 \to_1 s_2 \wedge s_2 \to_2 s_3\}$.

We define reduction contexts and weak reduction contexts:

Definition 2.7. *A reduction context R is any context, such that its hole will be labeled with* sub *or* top *by the labeling algorithm in Fig. 2. A* weak reduction context*, R^-, is a reduction context, where the hole is not within a* `letrec`*-expression.*

Clearly, every reduction context is also a top context, and thus also a surface context.

Definition 2.8. *An expression s is a* weak head normal form (WHNF)*, if s is a value, or s is of the form* `letrec` *Env* `in` *v, where v is a value, or s is of the form* `letrec` *$x_1 = (c\ \overrightarrow{t}), \{x_i = x_{i-1}\}_{i=2}^{m}$, Env* `in` *$x_m$.*

By inspecting the labeling algorithm and the reduction rules one can verify that every WHNF is irreducible w.r.t. normal order reduction. As usual for lazy functional programming languages, we consider WHNFs as successfully evaluated programs and define the notion or convergence accordingly:

Definition 2.9. *An expression s* converges*, denoted as $s{\downarrow}$, iff there exists a WHNF t s.t. $s \xrightarrow{no,*} t$. This may also be denoted as $s \downarrow t$. We write $s{\uparrow}$ iff $s{\downarrow}$ does not hold.*

2.2. Program Transformations

A *program transformation* P is a binary relation on expressions. We write $s \xrightarrow{P} t$, if $(s,t) \in P$. For a set of contexts X and a transformation P, the transformation (X, P) is the closure of P w.r.t. the contexts in X, i.e. $C[s] \xrightarrow{X,P} C[t]$ iff $C \in X$ and $s \xrightarrow{P} t$.

Ignoring the label, the reduction rules in Fig. 3 are also program transformations. In Fig. 4 additional program transformations are defined. The transformation (gc) performs garbage collection by removing unused `letrec`-environments, and (cpx), (cpax) copy variables, and can be used to shorten chains of indirections. The transformation (cpcx) copies a constructor application into a referenced position, where the arguments are shared by new `letrec`-bindings. Similarly, (abs) and (abse) perform this sharing without copying the constructor application. The transformation (ucp) means "unique copying" and it inlines a shared expression which is referenced only once.

Definition 2.10. *We define unions for the rules in Fig. 3: (case) is the union of (case-c), (case-in), (case-e); (seq) is the union of (seq-c), (seq-in), (seq-e); (cp) is the union of (cp-in), (cp-e); (llet) is the union of (llet-in), (llet-e); and (lll) is the union of (llet), (lapp), (lcase), and (lseq).*

We use the following unions for the transformations in Fig. 4: (gc) is the union of (gc1) and (gc2); (cpx) is the union of (cpx-in) and (cpx-e); (cpcx) is the union of (cpcx-in) and (cpcx-e); and (ucp) is the union of (ucp1), (ucp2), and (ucp3).

We use the unions of the rules also for normal order reduction and for instance write $\xrightarrow{no,llet}$ for $\xrightarrow{no,llet-in \vee no,llet-e}$.

2.3. Contextual Equivalence

As program equivalence we use contextual equivalence which equates two expressions if exchanging one program by the other program in any surrounding program context does not change the termination behavior. Due to the quantification over all contexts, it is sufficient to observe convergence.

Definition 2.11. *Let s, t be two* LR*-expressions. We define* contextual equivalence *$\sim_c$ w.r.t. the operational semantics of* LR*: Let $s \sim_c t$, iff for all contexts $C[\cdot]$: $C[s]{\downarrow} \iff C[t]{\downarrow}$.*

Contextual equivalence is a congruence, i.e. it is an equivalence relation which is compatible with contexts. Usually, proving two expressions to be contextually equivalent is difficult, since all contexts have to be taken into account. In contrast, disproving an equivalence is often easy, since a single counterexample is sufficient. E.g., the constants `True` and `False` are not contextually equivalent, since the context $\texttt{case}_{\texttt{Bool}}\ [\cdot]\ (\texttt{True} \texttt{->} \texttt{True})\ (\texttt{False} \texttt{->} \Omega)$ distinguishes them; where Ω is a divergent expression, for instance $\Omega = (\lambda x.x\ x)\ (\lambda y.y\ y)$.

Definition 2.12. *A program transformation P is* correct*, if it preserves contextual equivalence, i.e. $P \subseteq \sim_c$.*

In [24] we proved that all introduced transformations are correct:

Proposition 2.13 ([24])**.** *The program transformations (lbeta), (case), (seq), (cp), (lll), (gc), (cpx), (cpax), (cpcx), (abs), (abse), (xch), and (ucp) are correct.*

3. Improvement in the LR-Calculus

While contextual equivalence is a correctness criterion for program transformations, it has no requirements on the transformation being an *optimization* w.r.t. time (or space) complexity of a program. This is where the improvement relation comes into play and restricts contextual equivalence of s and t by the additional requirement that s may be replaced by t (within a program) if the number of computation steps for successfully evaluating the whole program is not increased. We define measures for estimating the time consumption: on the one hand we count all reduction steps, on the other hand we count essential reduction steps where we use a generic approach and leave some freedom in the choice of which steps are counted:

Definition 3.1. *In the following let* $\mathfrak{A} := \{(lbeta), (case), (seq)\}$ *and* $\emptyset \neq A \subseteq \mathfrak{A}$. *Let* t *be a closed expression with* $t{\downarrow}t_0$. *Then* $\mathtt{rln}_A(t)$ *is the number of* a*-reductions with* $a \in A$ *in the normal order reduction* $t{\downarrow}t_0$. *With* $\mathtt{rlnall}(t)$ *we denote the number of all reductions in* $t{\downarrow}t_0$. *The measures are defined as* ∞, *if* $t{\uparrow}$, *since we are only interested in values on converging expressions.*

Similarly, we apply the measures to reduction sequences $t \xrightarrow{*} t'$.

The restriction of counting mainly reductions in $\mathfrak{A}$ can be justified as follows: The (cp)-reductions are not counted, since the number of (cp)-reductions of an expression t is at most $2 \cdot \mathtt{rln}_{\mathfrak{A}}(t) + 1$: every (no,cp)- is followed by an (no,lbeta)- or an (no,seq)-reduction, or it is the last reduction.

Also (lll)-reductions are not counted, since these can be more efficiently implemented on abstract machines, often more efficient than in the calculus model, by floating environments to the top in one step instead of doing it step-by-step. A further deviation from real run-time is the size of the abstractions, which are duplicated in a (cp)-reduction. Also the search for a redex (modeled by our labeling algorithm) is not counted by our measures (which is different from [10]). If the computation is long compared to the size of the expression, then the sizes of abstractions can be considered as constant. In particular, in call-by-need computation, the size of abstractions cannot be increased. This alleviates the error made by not counting the size (see also Theorem 5.21).

Using measures with $A \neq \mathfrak{A}$ has two motivations: On the one hand we get more exact information, and on the other hand, we can show an improvement relation also for variations of the measure like $a_1 \cdot \mathtt{rln}_{\{lbeta\}} + a_2 \cdot \mathtt{rln}_{\{case\}} + a_3 \cdot \mathtt{rln}_{\{seq\}}$ with $a_i \geq 0$.

We analyze the relative number of (lbeta)-reductions w.r.t. other reductions in $\mathfrak{A}$. First we define the size of an expression:

Definition 3.2. *For an* LR*-expression* s, *its size* $\mathtt{size}(s)$ *is defined as*

$$
\begin{array}{lcl}
\mathtt{size}(x) & := & 1, \text{ if } x \in Var \\
\mathtt{size}(\lambda x.s) & := & 1 + \mathtt{size}(s) \\
\mathtt{size}(s\ t) & := & 1 + \mathtt{size}(s) + \mathtt{size}(t) \\
\mathtt{size}(\mathtt{seq}\ s\ t) & := & 1 + \mathtt{size}(s) + \mathtt{size}(t) \\
\mathtt{size}(\mathtt{letrec}\ x_1 = s_1 \ldots, x_n = s_n\ \mathtt{in}\ s_0) & := & 1 + \sum_{i=0}^{n} \mathtt{size}(s_i) \\
\mathtt{size}(\mathtt{case}_K\ t\ (pat_1\ \mathtt{->} s_1) \ldots (pat_{ar(K)}\ \mathtt{->} s_{ar(K)})) & := & 1 + \mathtt{size}(t) + \sum_{i=0}^{ar(K)} (\mathtt{size}(pat_i) + \mathtt{size}(s_i)) \\
\mathtt{size}(c_{K,i}\ s_1\ \ldots\ s_{ar(c_{K,i})}) & := & 1 + \sum_{i=1}^{ar(c_{K,i})} \mathtt{size}(s_i)
\end{array}
$$

Proposition 3.3. *Let* s *be a closed expression. Then* $\mathtt{rln}_{\mathfrak{A}}(s) \leq (\mathtt{size}(s) + 1) \cdot (\mathtt{rln}_{\{lbeta\}}(s) + 1)$.

Proof. It suffices to assume that $s{\downarrow}$, since otherwise the claim obviously holds. So suppose $s \xrightarrow{no,*} t$ where t is a WHNF. For every (cp)-reduction in this sequence exactly one of the following properties holds: i) The (cp)-reduction is the last reduction in the sequence. ii) The (cp)-reduction is followed by a (seq)-reduction (which removes the copied abstraction). iii) The (cp)-reduction is followed by an $(lbeta)$-reduction (which uses the copied abstraction in function position). Let us conflate these reductions, i.e. we write $(cpseq)$ for

a (cp)-reduction followed by (seq), ($cplbeta$) for a (cp)-reduction followed by ($lbeta$), and ($cpWHNF$) for a (cp)-reduction as a last reduction in a normal order reduction to a WHNF. Let $\mathtt{size}_{ni}$ be the size of an expression without counting indirections ($x = y$-bindings), i.e. $\mathtt{size}_{ni}(s) = \mathtt{size}(s) - \mathit{indirections}(s)$, where $\mathit{indirections}(s)$ is the number of $\mathtt{letrec}$-bindings $x = y$ in s where y is a variable.

Then all the reduction rules of LR different from ($cplbeta$) and ($cpWHNF$) do not increase the $\mathtt{size}_{ni}$, and the reduction rules ($case$), (seq), and ($cpseq$) strictly decrease $\mathtt{size}_{ni}$.

Thus to estimate the number of ($case$)-, (seq)- and ($cpseq$)-reductions it is sufficient to determine the initial size of the expression and the size increase of ($cplbeta$)-reductions.

In the reduction sequence $s \xrightarrow{no,*} t$, a single ($cplbeta$)-reduction increases the $\mathtt{size}_{ni}$ at most by $\mathtt{size}(s)$. Thus the overall size which can be used by (seq), ($cpseq$), and ($case$) is (1+number of ($cplbeta$)) $\cdot$ ($\mathtt{size}(s)$) (the size increase of ($cpWHNF$) cannot be used by (seq), ($cpseq$), or ($case$)).

The number of ($cplbeta$)-reductions in $s \xrightarrow{no,*} t$ is at most $\mathtt{rln}_{\{lbeta\}}(s)$. Hence $\mathtt{rln}_{\mathfrak{A}}(s)$ is at most $\mathtt{rln}_{\{lbeta\}}(s)$ plus $\mathtt{size}(s) \cdot (\mathtt{rln}_{\{lbeta\}}(s) + 1)$, which is at most $(\mathtt{size}(s) + 1) \cdot (\mathtt{rln}_{\{lbeta\}}(s) + 1)$. □

Remark 3.4. *Note that there are normal-order reduction sequences with an arbitrary number of ($lbeta$)-reductions, but without any ($case$)- or (seq)-reductions, hence Proposition 3.3 is not valid for $\{case\}$, $\{seq\}$ or $\{case, seq\}$ instead of $\{lbeta\}$.*
As a concrete example, let us consider the following expressions t_n, for $n \geq 0$ defined by $t_n := \mathtt{letrec}\ \{x_i = \lambda y.((x_{i-1}\ x_0)\ (x_{i-1}\ x_0))\}_{i=2}^{n}, x_0 = \lambda z.z\ \mathtt{in}\ (x_n\ x_0)$. Evaluation of t_n requires $3 \cdot 2^n - 2$ (no,lbeta)-reductions, but neither any (no,seq)- nor (no,case)-reduction. The size of t_n is $\mathtt{size}(t_n) = 8 \cdot n + 6$. Thus, for any set A with $\emptyset \neq A \subseteq \{case, seq\}$, in particular $\mathtt{rln}_{\{lbeta\}} = \mathtt{rln}_{\mathfrak{A}}(t_n) = 3 \cdot 2^n - 2 > (\mathtt{size}(t_n) + 1) \cdot (\mathtt{rln}_A(t_n) + 1) = 8 \cdot n + 7$ for all $n > 3$, and furthermore, for any polynomial poly: $\mathtt{rln}_{\mathfrak{A}}(t_n) > poly(\mathtt{size}(t_n), \mathtt{rln}_A(t_n))$ for all $n \geq n_0$ for some n_0.

We present results from [24, 22] w.r.t. reduction lengths of the transformations in Figs. 3 and 4, which are slightly more general, and can be derived from these papers.

Theorem 3.5 ([24]). *Let t be a closed* LR*-expression s.t. $t{\downarrow}t_0$.*

1. *If $t \xrightarrow{C,a} t'$ and $a \in \{case, seq, lbeta\}$, then for all A: $\mathtt{rln}_A(t) \geq \mathtt{rln}_A(t')$ and $\mathtt{rlnall}(t) \geq \mathtt{rlnall}(t')$.*
2. *If $t \xrightarrow{C,cp} t'$, then for all A: $\mathtt{rln}_A(t) = \mathtt{rln}_A(t')$.*
3. *If $t \xrightarrow{S,a} t'$ and $a \in \{case, seq, lbeta\}$, then for all A: $\mathtt{rln}_A(t) \geq \mathtt{rln}_A(t') \geq \mathtt{rln}_A(t) - 1$ if $a \in A$, and otherwise, if $a \notin A$, then $\mathtt{rln}_A(t) = \mathtt{rln}_A(t')$.*
4. *If $t \xrightarrow{C,a} t'$ and $a \in \{lll, gc\}$, then for all A: $\mathtt{rln}_A(t) = \mathtt{rln}_A(t')$ and $\mathtt{rlnall}(t) \geq \mathtt{rlnall}(t')$. For $a = gc1$ the equation $\mathtt{rlnall}(t) = \mathtt{rlnall}(t')$ holds.*
5. *If $t \xrightarrow{C,a} t'$ and $a \in \{cpx, cpax, xch, cpcx, abs\}$, then for all A: $\mathtt{rln}_A(t) = \mathtt{rln}_A(t')$ and $\mathtt{rlnall}(t) = \mathtt{rlnall}(t')$.*
6. *If $t \xrightarrow{C,ucp} t'$, then for all A: $\mathtt{rln}_A(t) = \mathtt{rln}_A(t')$ and $\mathtt{rlnall}(t) \geq \mathtt{rlnall}(t')$.*

3.1. The Improvement Relation

The improvement relation relates only contextual equivalent expressions and requires that the reduction length $\mathtt{rln}_A(\cdot)$ is not increased:

Definition 3.6 (Improvement Relation). *For $s, t \in Expr$, $A \subseteq \mathfrak{A}$, let $s \preceq_A t$ (t is A-improved by s), iff $s \sim_c t$ and for all contexts $C[\cdot]$ s.t. $C[s]$, $C[t]$ are closed: $\mathtt{rln}_A(C[s]) \leq \mathtt{rln}_A(C[t])$. We write $t \succeq_A s$ if $s \preceq_A t$ holds. If $s \preceq_A t$ and $s \succeq_A t$, we write $s \approx_A t$.*

A program transformation P is an A-improvement *iff $P \subseteq \succeq_A$.*

Remark 3.7. *In practical applications of improvements (e.g. if they are applied for program optimization), those improvements are interesting which* strictly decrease *the number of computations steps when they are applied. However, for the definition of the improvement relation it is not useful to replace "$\mathtt{rln}_A(C[s]) \leq$*

$\mathtt{rln}_A(C[t])$" *by* $\mathtt{rln}_A(C[s]) < \mathtt{rln}_A(C[t])$*: such a definition would be empty, since there are contexts* C *where the evaluation of* $C[r]$ *ignores the position of* r*. Thus, those contexts would refute the modified improvement property. Also an additional requirement of the existence of at least one context* C *s.t.* $\mathtt{rln}_A(C[s]) < \mathtt{rln}_A(C[t])$ *is not helpful, since then the improvement relation would no longer be a congruence (using the same arguments as before).*

Let $\xi \in \{\leq, =, \geq\}$ be a relation on non-negative integers. For a class of contexts X[3], $A \subseteq \mathfrak{A}$, let $s \bowtie_{\xi,X,A} t$ iff $s \sim_c t$ and for all X-contexts X, s.t. $X[s], X[t]$ are closed: $\mathtt{rln}_A(X[s]) \ \xi \ \mathtt{rln}_A(X[t])$. In particular, instantiating $\bowtie_{\xi,X,A}$ with all contexts gives the already introduced improvement relations, i.e. $\bowtie_{\leq,C,A} = \preceq_A$, $\bowtie_{\geq,C,A} = \succeq_A$, and $\bowtie_{=,C,A} = \approx_A$.

The context lemma for improvement shows that it suffices to take reduction contexts into account for proving improvement. Its proof uses a generalized claim with multicontexts and it is similar to the ones for context lemmas for contextual equivalence in call-by-need lambda calculi (see [24, 18]).

Lemma 3.8 (Context Lemma for Improvement)**.** *Let* s, t *be expressions,* $\xi \in \{\leq, =, \geq\}$*, and* $A \subseteq \mathfrak{A}$*. Then* $s \ \bowtie_{\xi,R,A} \ t$ *(or* $s \ \bowtie_{\xi,T,A} \ t$*, or* $s \ \bowtie_{\xi,S,A} \ t$*) iff* $s \ \bowtie_{\xi,C,A} \ t$*.*

Proof. It suffices to consider the case for reduction contexts, since reduction contexts are also T- or S-contexts and thus if the context lemma holds for $\bowtie_{\xi,R,A}$ then it also holds for $\bowtie_{\xi,T,A}$ or $\bowtie_{\xi,S,A}$. One direction of the claim is trivial. For the other direction we prove a more general claim:

> *For all* $n \geq 0$ *and for all* $i = 1, \ldots, n$ *let* s_i, t_i *be expressions with* $s_i \sim_c t_i$ *and* $s_i \ \bowtie_{\xi,R,A} \ t_i$*. Then for all multicontexts* M *with* n *holes such that* $M[s_1, \ldots, s_n]$ *and* $M[t_1, \ldots, t_n]$ *are closed:* $\mathtt{rln}_A(M[s_1, \ldots, s_n]) \ \xi \ \mathtt{rln}_A(M[t_1, \ldots, t_n])$.

The proof is by induction on the pair (k, k') where k is the number of normal order reductions of $M[s_1, \ldots, s_n]$ to a WHNF, and k' is the number of holes of M. If M (without holes) is a WHNF, then the claim holds. If $M[s_1, \ldots, s_n]$ is a WHNF, and no hole is in a reduction context, then also $M[t_1, \ldots, t_n]$ is a WHNF and $\mathtt{rln}_A(M[s_1, \ldots, s_n]) = 0 = \mathtt{rln}_A(M[t_1, \ldots, t_n])$.

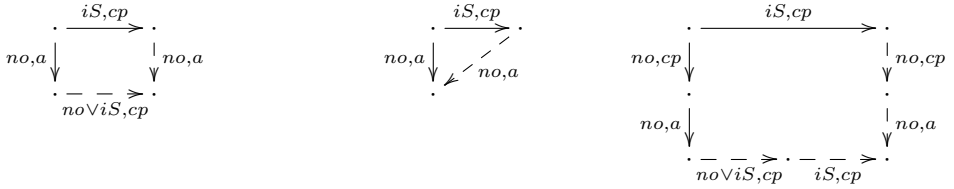
If in $M[s_1, \ldots, s_n]$ one s_j is in a reduction context, then one hole, say i, of M is in a reduction context and $M[t_1, \ldots, t_{i-1}, \cdot, t_{i+1}, \ldots, t_n]$ is a reduction context. Applying the induction hypothesis to the multicontext $M[\ldots, \cdot, s_i, \cdot, \ldots]$ (with $n-1$ holes) shows $\mathtt{rln}_A(M[s_1, \ldots, s_{i-1}, \ s_i, s_{i+1}, \ldots, s_n]) \ \ \xi \ \ \mathtt{rln}_A(M[t_1, \ldots, t_{i-1}, \ s_i, t_{i+1}, \ldots, t_n])$, and the assumption shows $\mathtt{rln}_A(M[t_1, \ldots, \ t_{i-1}, \ s_i, t_{i+1}, \ldots, t_n]) \ \ \xi \ \ \mathtt{rln}_A(M[t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ \ldots, t_n])$, and hence $\mathtt{rln}_A(M[s_1, \ldots, s_n]) \ \ \xi \ \ \mathtt{rln}_A(M[t_1, \ldots, t_n])$.

If in $M[s_1, \ldots, s_n]$ there is no s_j in a reduction context, then the first normal order reduction is of the form $M[s_1, \ldots, s_n] \xrightarrow{no,a} M'[s'_1, \ldots, s'_{n'}]$, and it may copy or shift some of the s_i where $s'_j = \rho(s_i)$ for some variable permutation ρ. However, the reduction type is the same for the first step of $M[s_1, \ldots, s_n]$ and $M[t_1, \ldots, t_n]$, i.e. $M[t_1, \ldots, t_n] \xrightarrow{no,a} M'[t'_1, \ldots, t'_{n'}]$ with $(s'_j, t'_j) = (\rho(s_i), \rho(t_i))$. We take for granted that the renaming can be carried through. The $\mathtt{rln}_A(.)$-contribution of both $\xrightarrow{no,a}$-reduction steps is either $m = 0$ or $m = 1$, depending on a. We can apply the induction hypothesis to $M'[s'_1, \ldots, s'_{n'}]$ and $M[t'_1, \ldots, t'_{n'}]$, and thus $\mathtt{rln}_A(M[s_1, \ldots, s_n]) = (m + \mathtt{rln}_A(M'[s'_1, \ldots, s'_{n'}])) \ \xi \ (m + \mathtt{rln}_A(M'[t'_1, \ldots, t'_{n'}])) = \mathtt{rln}_A(M[t_1, \ldots, t_n])$. □

3.2. Proof Technique

We explain our proof technique for proving that a program transformation is an improvement. The method is similar to the diagram-based technique used to show correctness of program transformations (see e.g. [24]). Let $\xrightarrow{P}$ be a correct program transformation. Due to the context lemma for improvement, it suffices to prove $\xrightarrow{P} \subseteq \bowtie_{\geq,X,A}$ (where X is instantiated with an appropriate set of contexts, i.e. reduction contexts R, top-contexts T, surface contexts S) to conclude $\xrightarrow{P} \subseteq \succeq_A$. To ease notation we will show this claim by proving that for all s, t with $s \xrightarrow{X,P} t$ the inequation $\mathtt{rln}_A(s) \geq \mathtt{rln}_A(t)$ holds, i.e. we use

[3] We will instantiate X with all contexts C; all reduction contexts R; all surface contexts S; or all top-contexts T



$a \in \{(lbeta), (case), (seq), (lll)\}$ $a \in \{(lbeta), (case), (seq)\}$ $a \in \{(lbeta), (seq)\}$

(1) (2) (3)

Figure 6: Forking diagrams for (iS,cp)

the closure of $\xrightarrow{P}$ w.r.t. X-contexts. If $\mathtt{rln}_A(s) = \infty$, then $s \sim_c t$ implies that $\mathtt{rln}_A(t) = \infty$ must also hold. Thus, the remaining case is that $s{\downarrow}$. Given the normal order reduction sequence from s to a WHNF, we construct a converging normal order reduction sequence for t and use this construction to prove that $\mathtt{rln}_A(s) \geq \mathtt{rln}_A(t)$ holds. For the construction of the reduction sequence, we use so-called sets of *forking diagrams* for the transformation $\xrightarrow{X,P}$ and often also additional forking diagrams for other transformations. These diagrams are then used in an inductive proof to construct the normal order reduction sequence for t, where often specific induction measures are required (if the diagrams are sufficiently complex).

A forking diagram describes how an overlap (a fork) of the form $s' \xleftarrow{no,+} s \xrightarrow{X,P} t$ can be closed by a sequence of reductions and transformations of the form $s' \xrightarrow{P',*} t' \xleftarrow{no,+} t$, where $\xrightarrow{P'}$ are program transformations, for instance $\xrightarrow{X,P}$, but it maybe also another transformation. A forking diagram abstracts from the concrete terms and thus it is written as depicted in Fig. 5, where often the reductions and transformations are more concrete (i.e. which rule of a normal order reduction is used). Sometimes also meta-terms are used to represent the expressions. The solid arrows are the given reductions (the fork) and the dashed arrows are the existentially quantified reductions. A set of forking diagrams is *complete* for transformation $\xrightarrow{X,P}$, if the set contains an applicable diagram for every fork $s' \xleftarrow{no,+} s \xrightarrow{X,P} t$, where a diagram is applicable if the reductions and transformations concretely exist. A complete set of forking diagrams is obtained by a case analysis which considers all overlaps of a transformation and a normal order reduction. This usually is a tedious task, and thus we do not include the case analyses in this paper, but they can be found in the technical report [22] or in [24].

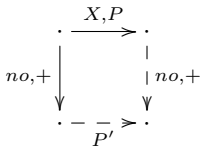


Figure 5: Forking diagram

If $s \xrightarrow{X,P} t$ and $s{\downarrow}$, i.e. $s = s_0 \xrightarrow{no} \cdots \xrightarrow{no} s_n$, where s_n is a WHNF, then a complete set of forking diagrams can be used to construct a reduction sequence which witnesses $t{\downarrow}$ and satisfies $\mathtt{rln}_A(s) \geq \mathtt{rln}_A(t)$: A single forking diagram is applied for every step $s_i \to s_{i+1}$ and one has to show hat the construction terminates (usually using an inductive argument). If other transformations than $\xrightarrow{X,P}$ occur in the forking diagrams, then usually also the corresponding sets of diagrams are required for the construction of the reduction sequences.

Note that the same technique can also be used to show $\xrightarrow{P} \subseteq \preceq$ or also $\xrightarrow{P} \subseteq \approx$.

3.3. Properties of (cp) and Other Transformations

We apply the diagram proof technique to prove properties of the (cp)-reduction using the diagrams in [24]. The set of forking diagrams in Fig. 6 covers all cases of overlaps between a normal order reduction and an $\xrightarrow{iS,cp}$-transformation where iS means the closure of (cp) in surface contexts, but excluding (no,cp) reductions. The diagrams are obtained from Lemmas B.8 and B.9 of the appendix of [24]. We use these diagrams to prove the following theorem.

Theorem 3.9. *Let t be closed s.t. $t{\downarrow}$. If $t \xrightarrow{C,\ cp} t'$ then $\mathtt{rln}_A(t) = \mathtt{rln}_A(t')$.*

Proof. We use the context lemma 3.8 for improvements for the relation $\approx_A$ and for surface contexts, i.e., we show $(cp) \subseteq \bowtie_{=,S,A}$ to derive $\bowtie_{=,C,A} = \approx_A$. Let s be closed and $s \xrightarrow{S,cp} s'$. We already know that $s \sim_c s'$,

hence we can assume that $s\!\downarrow$, which implies $s'\!\downarrow$. We can also assume that the reduction is not normal order, since in this case the claim is trivial.

We prove $\mathtt{rln}_A(s) = \mathtt{rln}_A(s')$ by induction on $\mathtt{rln}_A(s)$ and then on the length of a normal order reduction. If the length is 0, then s is a WHNF, and hence s' is a WHNF.

If $s \xrightarrow{no,a} s_1$ for $a \in A$, then $\mathtt{rln}_A(s_1) = \mathtt{rln}_A(s) - 1$. Either diagram (1) or (2) holds. In the former case we can apply the induction hypothesis, and in the latter case the claim obviously holds. If $s \xrightarrow{no,cp} s_1$, then there are two cases: s_1 is a WHNF. In this case it is easy to see that there is a WHNF s_2 with $s' \xrightarrow{no,cp} s_2$, and the claim holds. The other case is that diagram (3) is applicable. Then $s_1 \xrightarrow{no,a} s_2$ and $\mathtt{rln}_A(s_2) = \mathtt{rln}_A(s) - 1$. Hence we can apply the induction hypothesis twice, and obtain the claim. If $s \xrightarrow{no,lll} s_1$, then diagram (1) applies, and by he induction hypothesis, we have $s' \xrightarrow{no,lll} s_1$, and since $\mathtt{rln}_A(s') = \mathtt{rln}_A(s_1)$, we obtain $\mathtt{rln}_A(s) = \mathtt{rln}_A(s')$. □

Due to the exact analyses in [24] on the influence of the reduction rules (Fig. 3) and the additional transformations (Fig. 4) concerning the reduction lengths as stated in Theorems 3.5 and 3.9, Proposition 2.13 and Lemma 3.8 imply the following theorem:

Theorem 3.10. *The transformations (case), (seq), (lbeta), (cp), (lll), (gc), (cpx), (cpax), (xch), (abs), and (ucp) are A-improvements. Moreover, for $b \in \{$(cp), (lll), (gc), (cpx), (cpax), (xch), (abs), (ucp)$\}$ the inclusion $b \subseteq \preceq_A$ and thus the inclusion $b \subseteq \approx_A$ holds.*

4. Common Subexpression Elimination

In this section, we show the improvement property for common subexpression elimination (cse):

(cse) $M[s,\ldots,s] \to \mathtt{letrec}\ x = s\ \mathtt{in}\ M[x,\ldots,x]$
where x is a fresh variable and the multicontext M does not capture a variable in s

Although it seems to be obvious that (cse) is an improvement, its proof is not trivial, due to several reasons. The calculus LR is call-by-need, which means that (parts of) computations can be shared. To the best of our knowledge, we are only aware of a correctness proof of (*cse*) in a call-by-need calculus via a translation into a call-by-name calculus on infinite trees [23], which cannot be used to analyze resource usage under call-by-need. The improvement-property of (cse) was mentioned as a conjecture in [10].

We describe the structure of our proof that $(cse) \subseteq \succeq_A$ holds for all A. As a first step, we consider the general-copy rule (gcp) which allows to inline an arbitrary expression[4]:

(gcp) $\mathtt{letrec}\ x = s\ \mathtt{in}\ C[x] \ \to\ \mathtt{letrec}\ x = s\ \mathtt{in}\ C[s]$. (where C does not capture x nor a variable in s)

We will show that the inverse of (*gcp*) (called (*pcg*) by reversing the name "gcp") is an improvement (and thus $(gcp) \subseteq \preceq_A$). This result together with the result that garbage collection is invariant w.r.t. $\approx_A$ (Theorem 3.10) imply that (cse) is an improvement, since

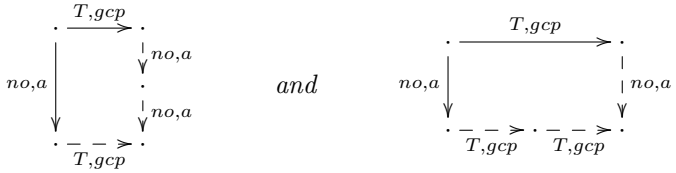
$$\mathtt{letrec}\ x = s\ \mathtt{in}\ M[x,\ldots,x] \xrightarrow{gcp,*} \mathtt{letrec}\ x = s\ \mathtt{in}\ M[s,\ldots,s] \xrightarrow{gc} M[s,\ldots,s] \tag{4}$$

(where (*gc*) is applicable, since x is fresh for M and s) and thus $(cse) \subseteq (\succeq_A \circ \approx_A)$ which implies $(cse) \subseteq \succeq_A$, since $\succeq_A \subseteq \approx_A$ and since $\succeq_A$ is a precongruence.

It remains to prove that the inverse of (*gcp*) is an improvement.

[4]Note that $x \in FV(s)$ is permitted in (*gcp*).

Remark 4.1. *The diagram based method fails to show correctness of* (gcp)*, since for proving that* $s \xrightarrow{T,gcp} t$*,* $s{\downarrow}$ *implies* $t{\downarrow}$*, diagrams of the shapes*



occur. For the combination of both diagrams no induction proof is possible, since the diagrams can be used to construct unbounded (but non-existing) reduction sequences for t *from a given finite sequence for* s*.*

Even though the diagram method fails for this problem, correctness of (gcp) was recently shown in [23] by another technique.

Using the correctness result we are able to prove that the transformation (pcg) – a slight extension of the inverse of (gcp) to letrec-environments – is an improvement by using a complete set of forking diagrams for (T, pcg) and then applying the context lemma for improvement.

However, the forking diagrams for (T, pcg) (shown in Fig. 8) contain further transformations. They include some additional transformations from Fig. 4 which are already analyzed w.r.t. their behavior on reduction lengths, but also a further transformation, called $(pcgE)$, which unions several variants of (pcg) in environments. As already mentioned, this forces us to develop a complete set of forking diagrams also for $(T, pcgE)$. Luckily, these diagrams (shown in Fig. 7) do not require new transformations and thus the complete sets of forking diagrams for (T, pcg) and $(T, pcgE)$ and the results on the additional transformations are sufficient to establish that (pcg) is an improvement.

Definition 4.2. *The transformation* $(pcgE)$ *is the union of the following rules, where we assume that the expression on the left hand sides fulfills the distinct variable convention,* $LV(Env) \cap FV(Env') = \emptyset$*, and there is a renaming of bound variables* $\alpha : LV(Env') \to LV(Env)$ *s.t. for* $Env = \{x_1 = s_1, \ldots, x_n = s_n\}$ *and* $Env'\alpha = \{x_1 = t_1, \ldots, x_n = t_n\}$ *the expressions* s_i *and* t_i *are alpha-equivalent for* $i = 1, \ldots, n$*.*

$(\text{pcgE}_1\text{-in})$ $\texttt{letrec}\ Env, Env_2\ \texttt{in}\ C[\texttt{letrec}\ Env', Env_3\ \texttt{in}\ r]$
$\to \texttt{letrec}\ Env, Env_2\ \texttt{in}\ C[\texttt{letrec}\ Env_3\alpha\ \texttt{in}\ r\alpha]$
$(\text{pcgE}_2\text{-in})$ $\texttt{letrec}\ Env, Env_2\ \texttt{in}\ C[\texttt{letrec}\ Env'\ \texttt{in}\ r] \to \texttt{letrec}\ Env, Env_2\ \texttt{in}\ C[r\alpha]$
$(\text{pcgE}_1\text{-e})$ $\texttt{letrec}\ Env, Env_2, x = C[\texttt{letrec}\ Env', Env_3\ \texttt{in}\ r]\ \texttt{in}\ s$
$\to \texttt{letrec}\ Env, Env_2, x = C[\texttt{letrec}\ Env_3\alpha\ \texttt{in}\ r\alpha]\ \texttt{in}\ s$
$(\text{pcgE}_2\text{-e})$ $\texttt{letrec}\ Env, Env_2, x = C[\texttt{letrec}\ Env'\ \texttt{in}\ r] \to \texttt{letrec}\ Env, Env_2, x = C[r\alpha]\ \texttt{in}\ s$
(pcgE_3) $\texttt{letrec}\ Env, Env', Env_3\ \texttt{in}\ r \to \texttt{letrec}\ Env, Env_3\alpha\ \texttt{in}\ r\alpha$

The transformation (pcg) *is the union of the rules:*

(pcg-in) $\texttt{letrec}\ x = s, Env\ \texttt{in}\ C[s] \to \texttt{letrec}\ x = s, Env\ \texttt{in}\ C[x]$
(pcg-e) $\texttt{letrec}\ x = s, Env, y = C[s]\ \texttt{in}\ r \to \texttt{letrec}\ x = s, Env, y = C[x]\ \texttt{in}\ r$

Note that the conditions on Env, Env' preclude capturing variables, and furthermore several conflicts cannot occur, like copying from Env into Env', or binding chains that intersect both Env, Env'. Correctness of the transformations (pcg) and $(pcgE)$ is a consequence of a result shown in [23]:

Proposition 4.3. *The program transformations* (pcg) *and* $(pcgE)$ *are correct.*

Proof. In [23] the following result for LR was obtained:

> *For an expression* s*, its infinite tree* $IT(s)$ *is derived by unfolding all* `letrec`*-bindings (and removing the* `letrec`*). If* $IT(s) = IT(t)$ *for expressions* s, t *(where* $=$ *is syntactic equality modulo* α*-equivalence on infinite trees), then* $s \sim_c t$*.*

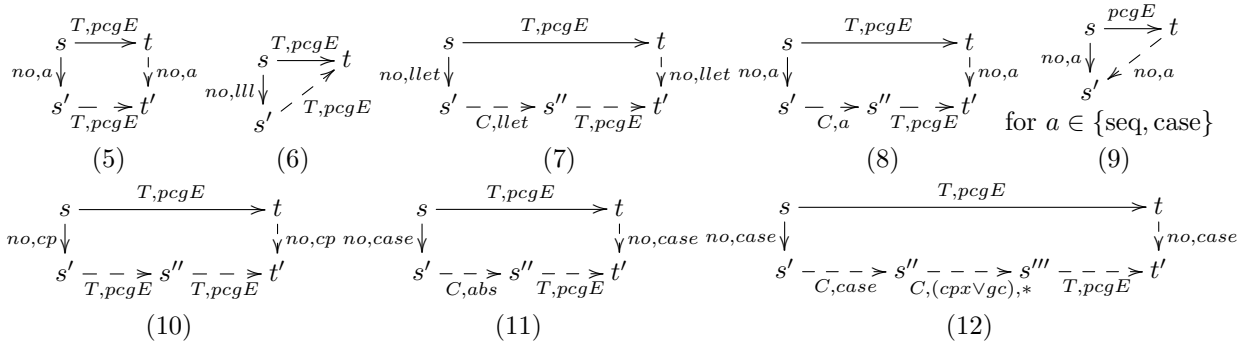


Figure 7: Forking diagrams for (pcgE)

Since $s \xrightarrow{pcg \vee pcgE} t$ implies $IT(s) = IT(t)$, correctness of (pcg) and (pcgE) holds. □

We first prove that $(pcgE)$ is an improvement in Proposition 4.5, and thereafter we use this result in Lemma 4.6 and Theorem 4.7 to show that (pcg) is an improvement.

Lemma 4.4. *If $s \xrightarrow{T,pcg} s'$ for a WHNF s, then either s' is a WHNF or $s' \xrightarrow{no,cp} s''$ where s'' is a WHNF.*

Proposition 4.5. *If $s \xrightarrow{T,pcgE} t$, then for all $A \subseteq \mathfrak{A}$: $\mathtt{rln}_A(s) \geq \mathtt{rln}_A(t)$ and $\mathtt{rlnall}(s) \geq \mathtt{rlnall}(t)$. Moreover, $(pcgE)$ is an improvement, i.e. $(pcgE) \subseteq \succeq_A$ for all A.*

Proof. Let $s' \xleftarrow{no,a} s \xrightarrow{T,pcgE} t$. All possible overlappings (forks) can be joined by one of the diagrams in Fig. 7 (details are in [22]). By induction on $\mathtt{rlnall}(s)$ we show that $\mathtt{rlnall}(t) \leq \mathtt{rlnall}(s)$ and $\mathtt{rln}_A(t) \leq \mathtt{rln}_A(s)$. If $\mathtt{rlnall}(s) = 0$ then s is a WHNF, and t must also be a WHNF and $\mathtt{rlnall}(t) = \mathtt{rln}_A(t) = 0$. If $\mathtt{rlnall}(s) > 0$, then let $s \xrightarrow{no,a} s'$.

- For diagram (5), we can apply the induction hypothesis to $s' \xrightarrow{T,pcgE} t'$, since $\mathtt{rlnall}(s') = \mathtt{rlnall}(s) - 1$. This shows that $\mathtt{rlnall}(t) \leq \mathtt{rlnall}(s)$ and $\mathtt{rln}_A(t) \leq \mathtt{rln}_A(s)$.
- For diagram (6), we apply the induction hypothesis to the transformation step $s' \xrightarrow{T,pcgE} t$ which shows $\mathtt{rlnall}(t) \leq \mathtt{rlnall}(s') = \mathtt{rlnall}(s) - 1$ and $\mathtt{rln}_A(t) \leq \mathtt{rln}_A(s') = \mathtt{rln}_A(s)$.
- For diagram (7), we have $\mathtt{rlnall}(s) > \mathtt{rlnall}(s')$ and $\mathtt{rln}_A(s) = \mathtt{rln}_A(s')$. By Theorem 3.5 (4) $\mathtt{rlnall}(s'') \leq \mathtt{rlnall}(s')$ and $\mathtt{rln}_A(s'') \leq \mathtt{rln}_A(s')$. We can apply the induction hypothesis to $s'' \xrightarrow{T,pcgE} t'$ which shows $\mathtt{rlnall}(t') \leq \mathtt{rlnall}(s'')$ and $\mathtt{rln}_A(t') \leq \mathtt{rln}(s'')$. This implies $\mathtt{rlnall}(t) \leq \mathtt{rlnall}(s)$ and also $\mathtt{rln}_A(t) \leq \mathtt{rln}_A(s)$.
- For diagram (8), we have $\mathtt{rlnall}(s') < \mathtt{rlnall}(s)$ and $\mathtt{rln}_A(s') \leq \mathtt{rln}_A(s)$. Theorem 3.5 shows that $\mathtt{rlnall}(s'') \leq \mathtt{rlnall}(s')$ and $\mathtt{rln}_A(s'') \leq \mathtt{rln}_A(s')$. We apply the induction hypothesis to $s'' \xrightarrow{T,pcgE} t'$ and have $\mathtt{rlnall}(t') \leq \mathtt{rlnall}(s'')$ and $\mathtt{rln}_A(t') \leq \mathtt{rln}_A(s'')$. This implies both $\mathtt{rlnall}(t) \leq \mathtt{rlnall}(s)$ as well as $\mathtt{rln}_A(t) \leq \mathtt{rln}_A(s)$.
- For diagram (9), obviously $\mathtt{rlnall}(s) = \mathtt{rlnall}(t)$ and $\mathtt{rln}_A(s) = \mathtt{rln}_A(t)$ hold.
- For diagram (10), we have $\mathtt{rlnall}(s') < \mathtt{rlnall}(s)$ and $\mathtt{rln}_A(s) = \mathtt{rln}_A(s')$. We apply the induction hypothesis to $s' \xrightarrow{T,pcgE} s''$ and have $\mathtt{rlnall}(s'') \leq \mathtt{rlnall}(s')$ and $\mathtt{rln}_A(s'') \leq \mathtt{rln}_A(s')$. Applying the induction hypothesis to $s'' \xrightarrow{T,pcgE} t'$ yields $\mathtt{rlnall}(t') \leq \mathtt{rlnall}(s'')$ and $\mathtt{rln}_A(t'') \leq \mathtt{rln}_A(s'')$. Since $t \xrightarrow{no,cp} t'$, this shows $\mathtt{rlnall}(t) \leq \mathtt{rlnall}(s)$ and $\mathtt{rln}_A(t) \leq \mathtt{rln}_A(s)$.

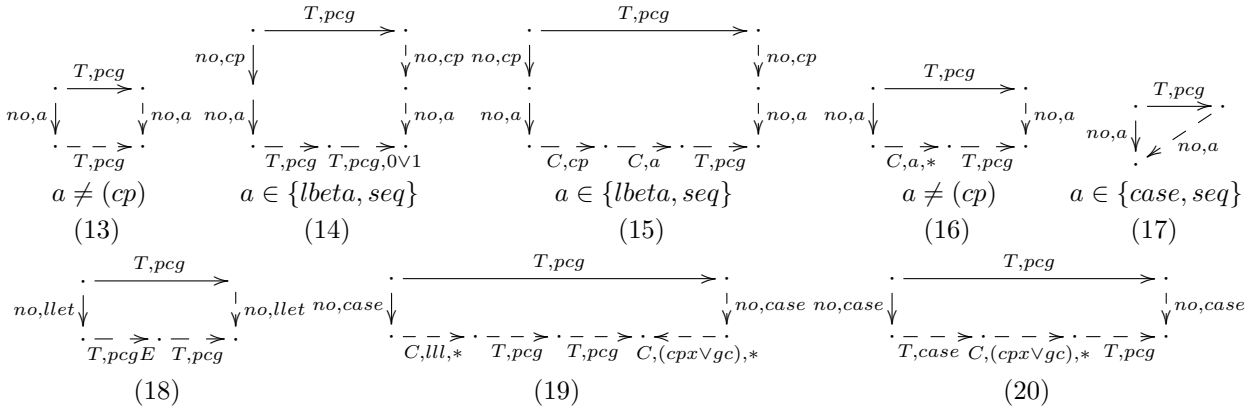


Figure 8: Forking-diagrams for (pcg)

- For diagram (11), we have $\texttt{rlnall}(s') < \texttt{rlnall}(s)$ and $\texttt{rln}_A(s) \leq \texttt{rln}_A(s')$. By Theorem 3.5 (5) we have $\texttt{rln}_A(s'') \leq \texttt{rln}_A(s')$ and $\texttt{rlnall}(s'') \leq \texttt{rlnall}(s')$. Applying the induction hypothesis to $s'' \xrightarrow{T,pcgE} t'$ yields $\texttt{rlnall}(t') \leq \texttt{rlnall}(s'')$ and $\texttt{rln}_A(t') \leq \texttt{rln}_A(s'')$ which shows $\texttt{rlnall}(t) \leq \texttt{rlnall}(s)$ and $\texttt{rln}_A(t) \leq \texttt{rln}_A(s)$.
- For diagram (12), we have $\texttt{rlnall}(s') < \texttt{rlnall}(s)$ and $\texttt{rln}_A(s) \leq \texttt{rln}_A(s')$. By Theorem 3.5 (1), (4), (5) we have $\texttt{rln}_A(s''') \leq \texttt{rln}_A(s')$ and $\texttt{rlnall}(s''') = \texttt{rlnall}(s')$. Applying the induction hypothesis to $s''' \xrightarrow{T,pcgE} t'$ yields $\texttt{rlnall}(t') \leq \texttt{rlnall}(s''')$ and $\texttt{rln}_A(t') \leq \texttt{rln}_A(s'')$ which shows $\texttt{rlnall}(t) \leq \texttt{rlnall}(s)$ and $\texttt{rln}_A(t) \leq \texttt{rln}_A(s)$.

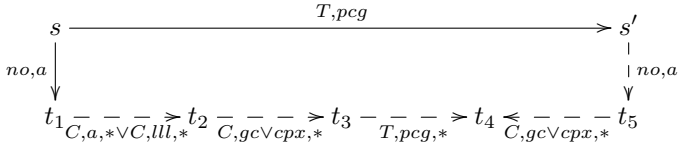
Since (pcgE) is correct (Proposition 4.3), Lemma 3.8 shows $(pcgE) \subseteq \succeq$. □

Lemma 4.6. *For closed s with $s \xrightarrow{T,pcg} s'$: $\texttt{rln}_A(s) \geq \texttt{rln}_A(s')$ for all A.*

Proof. Let $s{\downarrow}$, and $s \xrightarrow{T,pcg} s'$. Let $\emptyset \neq A \subseteq \mathfrak{A}$ be arbitrary but fixed. We show $\texttt{rln}_A(s) \geq \texttt{rln}_A(s')$ by induction on the measure $(\texttt{rln}_{\mathfrak{A}}(s), \texttt{rlnall}(s))$, ordered lexicographically. For the base case $\texttt{rln}_{\mathfrak{A}}(s) = \texttt{rlnall}(s) = 0$, the expression s is a WHNF and Lemma 4.4 shows that $\texttt{rln}_A(s') = 0$.

For the induction step, Fig. 8 shows the overlappings between normal order reduction-steps and a $\xrightarrow{T,pcg}$-transformation (for details see [22]). where the cases that the (T,pcg)-transformation is an inverse (C,cp)- or (C,cpx)-transformation are not covered, since in these cases Theorem 3.5 (5), or Theorem 3.9 show the claim. We consider the remaining cases:

1. If $s \xrightarrow{no,case \vee seq \vee lbeta} t_1$, then diagram (13), (16), (17), (19), or (20) of Fig. 8 holds. The diagrams can be summarized as follows where $a \in \{case, seq, lbeta\}$:



   We have $\texttt{rln}_{\mathfrak{A}}(t_1) = \texttt{rln}_{\mathfrak{A}}(s) - 1$ and by Theorem 3.5 we have $\texttt{rln}_{\mathfrak{A}}(t_3) \leq \texttt{rln}_{\mathfrak{A}}(t_1)$ and $\texttt{rln}_A(t_3) \leq \texttt{rln}_A(t_1)$. We apply the induction hypothesis for every step in $t_3 \xrightarrow{T,pcg,*} t_4$ and we derive $\texttt{rln}_{\mathfrak{A}}(t_4) \leq \texttt{rln}_{\mathfrak{A}}(t_1) < \texttt{rln}_{\mathfrak{A}}(s)$ and $\texttt{rln}_A(t_4) \leq \texttt{rln}_A(t_1)$. Theorem 3.5 shows that $\texttt{rln}_{\mathfrak{A}}(t_4) = \texttt{rln}_{\mathfrak{A}}(t_5)$ and $\texttt{rln}_A(t_4) = \texttt{rln}_A(t_5)$, and thus $\texttt{rln}_{\mathfrak{A}}(s') = \texttt{rln}_{\mathfrak{A}}(t_5) + 1 \leq \texttt{rln}_{\mathfrak{A}}(t_1) + 1 = \texttt{rln}_{\mathfrak{A}}(s)$ and either $\texttt{rln}_A(s') = \texttt{rln}_A(t_5) \leq \texttt{rln}_A(t_1) = \texttt{rln}_A(s)$ (if $a \notin A$), or $\texttt{rln}_A(s') = \texttt{rln}_A(t_5) + 1 \leq \texttt{rln}_A(t_1) + 1 = \texttt{rln}_A(s)$ (if $a \in A$).

2. Let $s \xrightarrow{no,cp} t_1$. If t_1 is a WHNF, then by Lemma 4.4 $s' \xrightarrow{no,cp,0\vee1} t_1'$ where t_1' is a WHNF and $\mathtt{rln}_{\mathfrak{A}}(s') \leq \mathtt{rln}_{\mathfrak{A}}(s)$ and also $\mathtt{rln}_A(s') \leq \mathtt{rln}_A(s)$ holds. If t_1 is not a WHNF, then $t_1 \xrightarrow{no,a} t_2$ where $a \in \{(lbeta), (seq)\}$ and diagram (14) or (15) of Fig. 8 holds. For diagram (14) we have:

$$\begin{array}{ccc} s & \xrightarrow{\quad T,pcg \quad} & s' \\ {\scriptstyle no,cp}\downarrow & & \downarrow{\scriptstyle no,cp} \\ t_1 & & t_1' \\ {\scriptstyle no,a}\downarrow & & \downarrow{\scriptstyle no,a} \\ t_2 & \xrightarrow[T,pcg]{} & t_3 \xrightarrow[T,pcg,0\vee1]{} t_4 \end{array}$$

Then $\mathtt{rln}_{\mathfrak{A}}(t_2) < \mathtt{rln}_{\mathfrak{A}}(s)$ and we apply the induction hypothesis to the transformation step $t_2 \xrightarrow{T,pcg} t_3$ which shows $\mathtt{rln}_{\mathfrak{A}}(t_3) \leq \mathtt{rln}_{\mathfrak{A}}(t_2) < \mathtt{rln}_{\mathfrak{A}}(s)$. We then apply the induction hypothesis to $t_3 \xrightarrow{T,pcg,0\vee1} t_4$ which shows $\mathtt{rln}_{\mathfrak{A}}(t_4) \leq \mathtt{rln}_{\mathfrak{A}}(t_2) < \mathtt{rln}_{\mathfrak{A}}(s)$, $\mathtt{rln}_{\mathfrak{A}}(s') \leq \mathtt{rln}_{\mathfrak{A}}(s)$, and either $\mathtt{rln}_A(s') = \mathtt{rln}_A(t_4) \leq \mathtt{rln}_A(t_2) = \mathtt{rln}_A(s)$ (if $a \notin A$) or $\mathtt{rln}_A(s') = \mathtt{rln}_A(t_4) + 1 \leq \mathtt{rln}_A(t_2) + 1 = \mathtt{rln}_A(s)$ (if $a \in A$). Similarly, in diagram (15) the situation is: $t_2 \xleftarrow{no,a} t_1 \xleftarrow{no,cp} s \xrightarrow{pcg} s' \xrightarrow{no,cp} t_1' \xrightarrow{no,a} t_5$, and $t_2 \xrightarrow{C,cp} t_3 \xrightarrow{C,a} t_4 \xrightarrow{T,pcg} t_5$. Then $\mathtt{rln}_{\mathfrak{A}}(t_2) < \mathtt{rln}_{\mathfrak{A}}(s)$, $\mathtt{rln}_A(t_2) \leq \mathtt{rln}_A(s)$ and by Theorem 3.5 (1) $\mathtt{rln}_{\mathfrak{A}}(t_4) \leq \mathtt{rln}_{\mathfrak{A}}(t_2)$, $\mathtt{rln}_A(t_4) \leq \mathtt{rln}_A(t_2)$ and we apply the induction hypothesis to $t_4 \xrightarrow{T,pcg} t_5$ and have $\mathtt{rln}_{\mathfrak{A}}(t_5) \leq \mathtt{rln}_{\mathfrak{A}}(t_4) < \mathtt{rln}_{\mathfrak{A}}(s)$ and $\mathtt{rln}_{\mathfrak{A}}(s') \leq \mathtt{rln}_{\mathfrak{A}}(s)$ and also $\mathtt{rln}_A(s') \leq \mathtt{rln}_A(s)$.

3. If $s \xrightarrow{no,lll} t_1$, then diagram (13), (16), or (18) of Fig. 8 holds, which can be summarized as follows:

$$\begin{array}{ccc} s & \xrightarrow{\quad T,pcg \quad} & s' \\ {\scriptstyle no,lll}\downarrow & & \downarrow{\scriptstyle no,lll} \\ t_1 & \xrightarrow[C,lll,*\vee T,pcgE]{} & t_2 \xrightarrow[pcg]{} t_3 \end{array}$$

Then $\mathtt{rln}_{\mathfrak{A}}(t_1) = \mathtt{rln}_{\mathfrak{A}}(s)$, $\mathtt{rln}_A(t_1) = \mathtt{rln}_A(s)$, and $\mathtt{rlnall}(t_1) = \mathtt{rlnall}(s) - 1$. Theorem 3.5 (4), and Proposition 4.5 show that $\mathtt{rln}_{\mathfrak{A}}(t_2) \leq \mathtt{rln}_{\mathfrak{A}}(t_1)$, $\mathtt{rln}_A(t_2) \leq \mathtt{rln}_A(t_1)$, and $\mathtt{rlnall}(t_2) \leq \mathtt{rlnall}(t_1)$. Thus we can apply the induction hypothesis to $t_2 \xrightarrow{T,pcg} t_3$ which yields $\mathtt{rln}_A(t_3) \leq \mathtt{rln}_A(t_2)$ (also for $A = \mathfrak{A}$). Since $s' \xrightarrow{no,lll} t_3$, we have $\mathtt{rln}_A(s') = \mathtt{rln}_A(t_3) \leq \mathtt{rln}_A(t_2) = \mathtt{rln}_A(t_1) = \mathtt{rln}_A(s)$ which shows the claim. □

Theorem 4.7. *The program transformations (pcg), (pcgE), and (cse) are improvements.*

Proof. For (*pcgE*) the claim is proved in Proposition 4.5 and for (*pcg*) this follows from Lemma 4.6, correctness of (*pcg*) (Proposition 4.3) and the context lemma for improvement (Lemma 3.8).

As demonstrated in Eq. (4), the transformation (*cse*) can be represented as a sequence $\xleftarrow{gc} \cdot \xrightarrow{pcg,*}$ and since $(gc) \subseteq \approx_A$ for all A by Theorem 3.10, this shows that (*cse*) is an improvement for all A. □

5. Abstract Machine Semantics and Analysis of Resource Measures

The goal of this section is to investigate the relationship between the measure $\mathtt{rln}(\cdot)$ and the counting measures used in [10, 5] for their improvement relations. The following results are obtained in this section:

- In Theorem 5.15 we show that $\mathtt{rln}_A(\cdot)$ coincides with the number of essential transition steps $\mathtt{mln}_A(\cdot)$ for all A, of the abstract machine of [10].
- In Theorem 5.19 we compare the number of all transitions steps (the measure used by [10]) with our measure and the measure used by [5].
- For normal forms under (lll)-, (gc)- and (cpax)-reductions, the measures counting all machine transitions and counting essential machine transitions are almost the same (see Theorem 5.24).
- We show that the complexity of first-order functions is the same under various measures of reductions.

To compare and relate the resource consumption of two program calculi, we define the notion of an *asymptotically resource-preserving translation.* Therefore, we use the O-notation as follows.

For a set of expressions E and functions $f, g : E \to \mathbb{N}$, we write $f \in O(g)$, if there is a constant $c > 0$, s.t. for all $e \in E$: $f(e) \leq c \cdot g(e)$.

Definition 5.1. *Let $\mathcal{K}_1 = (E_1, \sim_1, \mathtt{size}_1, \mu_1)$, $\mathcal{K}_2 = (E_2, \sim_2, \mathtt{size}_2, \mu_2)$ be two calculi with sets of expressions E_i, contextual equivalences $\sim_i$, size-measures $\mathtt{size}_i$ for expressions, and measures μ_i for reduction length of expressions. A translation $\xi : \mathcal{K}_1 \to \mathcal{K}_2$ is* size-preserving*, iff $\mathtt{size}_1(e) \in O(\mathtt{size}_2(\xi(e)))$ and ξ is fully abstract; i.e., for all $e, e' \in E_1$: $e \sim_1 e' \iff \xi(e) \sim_2 \xi(e')$. A translation $\xi : \mathcal{K}_1 \to \mathcal{K}_2$ is* asymptotically resource-preserving*, iff ξ is size-preserving and there exists an $n \in \mathbb{N}$ with $\mu_1(e_1) \in O(\mathtt{size}_2(\xi(e_1))^n \cdot (\mu_2(\xi(e_1)) + 1))$.*

In Theorem 5.21 we prove several results on asymptotic resource-preserving translations between the calculus LR and the abstract machine of [10] w.r.t. different measures for reduction lengths.

5.1. Abstract Machine Semantics

The abstract machine used by [10] extends Sestoft's abstract machine mark 1 [26] by handling case-expressions and data constructors. We recall this abstract machine and extend it slightly to also handle `seq`-expressions

Definition 5.2. *The syntax of* machine expressions *is the same as the syntax for* LR*-expressions except that argument positions are restricted to variables, i.e. in applications $(s\ t)$,* `seq`*-expressions $(\mathtt{seq}\ s\ t)$, and constructor applications $(c\ t_1 \ldots t_{ar(c)})$ the expressions t, t_i must be variables.*

We define a translation from LR-expressions into machine expressions:

Definition 5.3. *The translation ψ from* LR*-expressions into machine expressions is*

$$
\begin{array}{ll}
\psi(x) & := x, \quad \text{if } x \in \mathit{Var} \\
\psi(s\ t) & := \mathtt{letrec}\ x = \psi(t)\ \mathtt{in}\ (\psi(s)\ x) \\
\psi(\mathtt{seq}\ s\ t) & := \mathtt{letrec}\ x = \psi(t)\ \mathtt{in}\ (\mathtt{seq}\ \psi(s)\ x) \\
\psi(c\ s_1\ \ldots\ s_n) & := \mathtt{letrec}\ x_1 = \psi(s_1), \ldots, x_n = \psi(s_n)\ \mathtt{in}\ (c\ x_1\ \ldots\ x_n) \\
\psi(M[s_1, \ldots, s_n]) & := M[\psi(s_1), \ldots, \psi(s_n)], \\
& \quad \text{for all multicontexts } M \text{ of the forms } \mathtt{letrec}\ x_1 = [\cdot], \ldots, x_n = [\cdot]\ \mathtt{in}\ [\cdot],\ \lambda x.[\cdot], \\
& \quad \text{or } \mathtt{case}_K\ [\cdot]\ (pat_1\ \mathtt{->}[\cdot]) \ldots (pat_n\ \mathtt{->}[\cdot]).
\end{array}
$$

Lemma 5.4. *For $s \in \mathit{Expr}$, $\mathtt{size}(s) \leq \mathtt{size}(\psi(s)) \leq \max(3, 2 + \mathit{maxArityOfConstructors}(s)) \cdot \mathtt{size}(s)$ where $\mathit{maxArityOfConstructors}(s)$ is the maximal arity of constructors that occur in s.*

Proof. Since $\mathtt{size}(\psi(s\ t)) = \mathtt{size}(\mathtt{letrec}\ x = \psi(t)\ \mathtt{in}\ (\psi(s)\ x)) = 3 + \mathtt{size}(\psi(s)) + \mathtt{size}(\psi(t))$, the factor 3 is required. $\mathtt{size}(\psi(c\ s_1 \ldots s_n)) = \mathtt{size}(\mathtt{letrec}\ x_1 = \psi(t_1), \ldots, x_n = \psi(t_n)\ \mathtt{in}\ (c\ x_1 \ldots x_n)) = 2 + n + \sum_i \mathtt{size}(\psi(s_i))$, which has to be compared with $\mathtt{size}(c\ s_1 \ldots s_n)$, which is $1 + \sum \mathtt{size}(s_i)$, hence the factor $2 + n$ is required. So we can take the maximum for an estimation. □

Since $\psi(t) \xrightarrow{C,ucp,*} t$, Theorem 3.5 (6) implies:

Lemma 5.5. *For all closed $t \in \mathit{Expr}$ and all A: $\mathtt{rln}_A(t) = \mathtt{rln}_A(\psi(t))$.*

Definition 5.6. *A* state *Q of the machine is a tuple $\langle \Gamma \mid s \mid S \rangle$, where Γ is an environment of bindings (like a* `letrec`*-environment), s is a machine expression, and S is a stack, with entries $\#\mathsf{upd}(x)$, $\#\mathsf{app}(x)$, $\#\mathsf{seq}(x)$, $\#\mathsf{case}(alts)$ where x is a variable and $alts$ is a set of* `case`*-alternatives. We use list notation for the stack S. The transition rules of the machine are shown in Fig. 9. With (*Unwind*) we denote the union of (*Unwind1*), (*Unwind2*), and (*Unwind3*). The machine* starts *with $\langle \emptyset \mid s \mid [] \rangle$ for an expression s and an* accepting state *is of the form $\langle \Gamma \mid v \mid [] \rangle$ where v is a value (i.e. an abstraction or a constructor application). A machine state $\langle \Gamma \mid s \mid S \rangle$ is* reachable *iff there exists an expression t s.t. $\langle \emptyset \mid t \mid [] \rangle \xrightarrow{*} \langle \Gamma \mid s \mid S \rangle$.*

(Lookup)	$\langle \Gamma, x = s \mid x \mid S\rangle \rightharpoonup \langle \Gamma \mid s \mid \#\mathsf{upd}(x) : S\rangle$
(Update)	$\langle \Gamma \mid v \mid \#\mathsf{upd}(x) : S\rangle \rightharpoonup \langle \Gamma, x = v \mid v \mid S\rangle$ where v is a value ($v = \lambda x.s$ or $v = c\,\overrightarrow{y}$)
(Unwind1)	$\langle \Gamma \mid (s\ x) \mid S\rangle \rightharpoonup \langle \Gamma \mid s \mid \#\mathsf{app}(x) : S\rangle$
(Unwind2)	$\langle \Gamma \mid (\mathtt{seq}\ s\ x) \mid S\rangle \rightharpoonup \langle \Gamma \mid s \mid \#\mathsf{seq}(x) : S\rangle$
(Unwind3)	$\langle \Gamma \mid \mathtt{case}_K\ s\ alts \mid S\rangle \rightharpoonup \langle \Gamma \mid s \mid \#\mathsf{case}(alts) : S\rangle$
(Subst)	$\langle \Gamma \mid \lambda x.s \mid \#\mathsf{app}(y) : S\rangle \rightharpoonup \langle \Gamma \mid s[y/x] \mid S\rangle$
(Seq)	$\langle \Gamma \mid v \mid \#\mathsf{seq}(y) : S\rangle \rightharpoonup \langle \Gamma \mid y \mid S\rangle$ where v is a value ($v = \lambda x.s$ or $v = c\,\overrightarrow{y}$)
(Branch)	$\langle \Gamma \mid c_{i,K}\,\overrightarrow{x} \mid \#\mathsf{case}(\ldots\ (c_{i,K}\,\overrightarrow{y}\ \mathtt{->}\ t)\ldots) : S\rangle \rightharpoonup \langle \Gamma \mid t[\overrightarrow{x}/\overrightarrow{y}] \mid S\rangle$
(Letrec)	$\langle \Gamma \mid \mathtt{letrec}\ Env\ \mathtt{in}\ s \mid S\rangle \rightharpoonup \langle \Gamma, Env \mid s \mid S\rangle$

Figure 9: Machine transitions

We define a mapping ϕ from reachable machine states to machine expressions:

$$\begin{array}{lll}
\phi(\langle \Gamma \mid s \mid \#\mathsf{upd}(x) : S\rangle) := \phi(\langle \Gamma, x = s \mid x \mid S\rangle) & & \phi(\langle \Gamma \mid s \mid \#\mathsf{app}(x) : S\rangle) := \phi(\langle \Gamma \mid (s\ x) \mid S\rangle) \\
\phi(\langle \Gamma \mid s \mid \#\mathsf{seq}(x) : S\rangle) := \phi(\langle \Gamma \mid (\mathtt{seq}\ s\ x) \mid S\rangle) & & \phi(\langle \Gamma \mid s \mid \#\mathsf{case}(alts) : S\rangle) := \phi(\langle \Gamma \mid (\mathtt{case}\ s\ alts) \mid S\rangle) \\
\phi(\langle \Gamma \mid s \mid []\rangle) := \mathtt{letrec}\ \Gamma\ \mathtt{in}\ s & &
\end{array}$$

Note that $\phi(\langle \Gamma \mid v \mid []\rangle) = \mathtt{letrec}\ \Gamma\ \mathtt{in}\ v$ and thus accepting states are mapped to WHNFs. Let $\psi(lbeta) =$ Subst, $\psi(case) =$ Branch, and $\psi(seq) =$ Seq, and for $\emptyset \neq A \subseteq \{lbeta, case, seq\}$ let $\psi(A) := \{\psi(a) \mid a \in A\}$, in particular $\psi(\mathfrak{A}) = \{\text{Subst}, \text{Branch}, \text{Seq}\}$.

Definition 5.7. *Let s be a closed machine expression such that $\langle \emptyset \mid s \mid []\rangle \xrightarrow{n} Q$ where Q is an accepting state. Then $\mathtt{mlnall}(s) = n$, and $\mathtt{mln}_A(s)$ counts the number of the (*Subst*)-, (*Branch*)-, and (*Seq*)-steps which are in $\psi(A)$. The number $\mathtt{mlnlook}(s)$ counts all (*Lookup*)-transitions. If no such sequence exists for s, then $\mathtt{mlnall}(s) = \mathtt{mln}_{\mathfrak{A}}(s) = \mathtt{mln}_A(s) = \mathtt{mlnlook}(s) = \infty$. We use $\mathtt{mln}_A(\cdot), \mathtt{mln}_{\mathfrak{A}}(\cdot), \mathtt{mlnlook}(\cdot)$ with the same meaning also for reachable states Q of the machine.*

Note that the improvement theory in [10] is based on the measure $\mathtt{mlnall}(\cdot)$, whereas the measure in [5] is $\mathtt{mlnlook}(\cdot)$ (see the comparison Theorems 5.21 and 5.24). Note also that the complexity of reduction of expressions is adequately measured with $\mathtt{mlnall}$:

Proposition 5.8. *The number of steps k on a RAM executing the reduction of a machine expression s on the abstract machine is smaller than $a \cdot \mathtt{size}(s) \cdot \mathtt{mlnall}(s)$ where a is a constant independent of s. This is under the assumption that storing and accessing the store requires constant time.*

Proof. The transitions (Lookup), (Unwind), (Seq), and (Letrec) require constant time, and the transitions (Update), (Subst), and (Branch) require time $O(\mathtt{size}(s))$, since the size of abstractions and case-alternatives in the transition sequence starting with $\langle \emptyset \mid s \mid []\rangle$ is at most $\mathtt{size}(s)$. □

5.2. Relating the Essential Reduction Steps

In this section we show that for a machine expression s the number of essential transition steps coincides with the number of essential normal order reductions in LR, i.e. we show that $\mathtt{rln}_A(s) = \mathtt{mln}_A(s)$. With Lemma 5.5 this also implies that for every LR-expression s the equation $\mathtt{rln}_A(s) = \mathtt{mln}_A(\phi(s))$ holds.

Lemma 5.9. *For a reachable state Q with $Q \rightharpoonup Q'$, one of the following cases holds for $\phi(Q)$ and $\phi(Q')$:*

$$Q \xrightarrow{\text{Lookup}\vee\text{Unwind}} Q' \qquad \phi(Q) = \phi(Q')$$

$$Q \xrightarrow{\text{Branch}} Q' \qquad \phi(Q) \xrightarrow[no,case-c]{} s \xrightarrow[T,cpax,*]{} s' \xrightarrow[T,gc,*]{} \phi(Q')$$

$$Q \xrightarrow{\text{Update}} Q' \qquad \phi(Q) \xrightarrow[T,cp]{} \phi(Q')$$

$$Q \xrightarrow{\text{Letrec}} Q' \qquad \phi(Q) \xrightarrow[T,lll,*]{} \phi(Q')$$

$$Q \xrightarrow{\text{Seq}} Q' \qquad \phi(Q) \xrightarrow[no,seq]{} \phi(Q')$$

$$Q \xrightarrow{\text{Update}} Q' \qquad \phi(Q) \xrightarrow[T,cpcx]{} s \xrightarrow[T,cpx,*]{} s' \xrightarrow[T,gc,*]{} \phi(Q')$$

$$Q \xrightarrow{\text{Subst}} Q' \qquad \phi(Q) \xrightarrow[no,lbeta]{} s \xrightarrow[T,cpax]{} s' \xrightarrow[T,gc]{} \phi(Q')$$

(mo,cpv-e) $\texttt{letrec}\ Env, x = v^{\mathsf{sub}}, y = C[x^{\mathsf{sub}\vee\mathsf{nontarg}}]\ \texttt{in}\ s \to \texttt{letrec}\ Env, x = v, y = C[v]\ \texttt{in}\ s$
where v is a value

(mo,cpv-in) $\texttt{letrec}\ Env, x = v^{\mathsf{sub}}\ \texttt{in}\ C[x^{\mathsf{sub}}] \to \texttt{letrec}\ Env, x = v\ \texttt{in}\ C[v]$ where v is a value

(mo,β-var) $C[(\lambda x.s)^{\mathsf{sub}}\ y] \to C[s[y/x]]$

(mo,casecx) $C[\texttt{case}_K\ (c_{K,i}\ \overrightarrow{x})^{\mathsf{sub}}\ \ldots (c_{K,i}\ \overrightarrow{y}\ \texttt{->}\ s)\ldots] \to C[s[\overrightarrow{x}/\overrightarrow{y}]]$

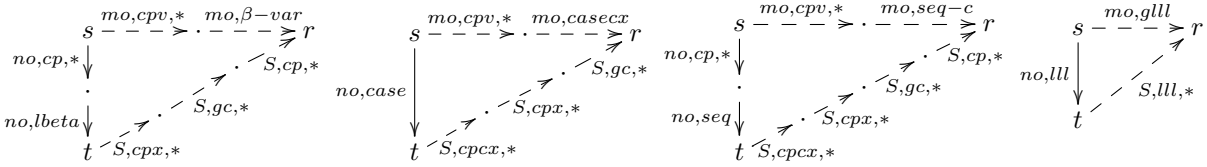
(mo,seq-c) $C[\texttt{seq}\ v^{\mathsf{sub}}\ x] \to C[x]$

(mo,glletm) $R^-[(\texttt{letrec}\ Env\ \texttt{in}\ s)^{\mathsf{sub}}] \to \texttt{letrec}\ Env\ \texttt{in}\ R^-[s]$

(mo,gllet-in) $\texttt{letrec}\ Env_1\ \texttt{in}\ C[(\texttt{letrec}\ Env_2\ \texttt{in}\ s)^{\mathsf{sub}}] \to \texttt{letrec}\ Env_1, Env_2\ \texttt{in}\ C[s]$

(mo,gllet-e) $\texttt{letrec}\ y = C[(\texttt{letrec}\ Env_1\ \texttt{in}\ s)^{\mathsf{sub}}], Env_2\ \texttt{in}\ t \to \texttt{letrec}\ y = C[s], Env_1, Env_2\ \texttt{in}\ t$

Figure 10: Machine order reduction rules



(21) (22) (23) (24)

Figure 11: Diagrams for transferring normal order reductions into machine order reductions

Proposition 5.10. *For a closed machine expression s and $\emptyset \neq A \subseteq \mathfrak{A}$, $\texttt{mln}_A(s) = n$ implies $\texttt{rln}_A(s) = n$.*

Proof. Let $\texttt{mln}_A(s) = n$. We consider the sequence of machine transitions from $\langle \emptyset \mid s \mid [] \rangle$ to an accepting state and construct a sequence of (no,lbeta)-, (no,case-c)-, (no,seq)-, (T,cp)-, (T,cpcx)-, (T,cpax)-, (T,lll)-, and (T-gc)-transformations from s to a WHNF.

So let $Q_0 = \langle \emptyset \mid s \mid [] \rangle \xrightarrow{k} Q_k$ where Q_k is an accepting state. We use induction on k: If $k = 0$ then $s = \phi(Q_0)$ is a WHNF. If $k > 0$ then we apply Lemma 5.9 to $Q_0 \rightharpoonup Q_1$ and then apply the induction hypothesis to $Q_1 \xrightarrow{k-1} Q_k$. This construction gives a sequence of transformations from $s = \phi(Q_0)$ to a WHNF, where the sum of (no,lbeta)-, (no,case-c)-, and (no,seq)-steps is n.

Now iteratively apply Theorems 3.5 and 3.9 from right to left to every transformation which is not a normal order reduction. Since all these steps leave the measure $\texttt{rln}_A(\cdot)$ unchanged, and the normal order step increases the measure by 1, this shows $\texttt{rln}_A(s) = n$. □

We define a variant of the normal order reduction for machine expressions – called *machine order reduction*: It uses the reduction rules shown in Fig. 10 and the machine order redex is found by the labeling algorithm in Definition 2.4. Let (mo,cpv) be the union of (mo,cpv-e) and (mo,cpv-in), and (mo,glll) be the union of (mo,glletm), (mo,gllet-in), and (mo,gllet-e). A *machine order WHNF* (MWHNF) is a value or an expression of the form $\texttt{letrec}\ Env\ \texttt{in}\ v$ where v is a value. Let $\overline{\psi}(lbeta) = (\text{mo},\beta\text{-var})$, $\overline{\psi}(case) = (mo, casecx)$, and $\overline{\psi}(seq) = (mo, seqc)$. For a closed expression s and $\emptyset \neq A \subseteq \mathfrak{A}$, let $\texttt{rln}_{mo,A}(s)$ be the number of (mo, a)-reductions with $(mo, a) \in \overline{\psi}(A)$ in a machine order reduction sequence from s to an MWHNF, and $\texttt{rln}_{mo,A}(s) = \infty$ otherwise.

Lemma 5.11. *If the machine expression s is a WHNF, then either s is also an MWHNF, or $s \xrightarrow{mo,cpv,*} s'$ where s' is an MWHNF.*

Lemma 5.12. *Let $s \xrightarrow{no,a} t$ where a is not a (cp), or $s \xrightarrow{no,cp,*} s' \xrightarrow{no,seq\vee lbeta} t$. The diagrams in Fig. 11 show how at least one machine order reduction can be performed for s, s.t. $s \xrightarrow{mo,+} r$ and how t and r are joinable by program transformations.*

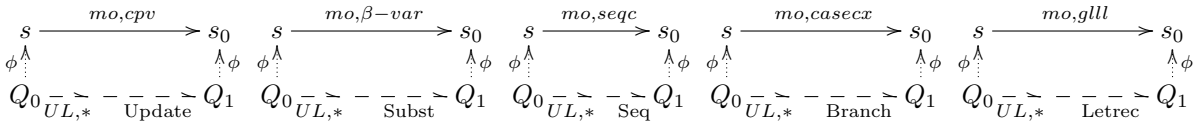
Proposition 5.13. *For all $\emptyset \neq A \subseteq \mathfrak{A}$, closed machine expressions s: $\texttt{rln}_A(s) = n \implies \texttt{rln}_{mo,A}(s) = n$.*

Proof. Let $s \xrightarrow{no,k} t_k$ where t_k is a WHNF. We use induction on $(\mathtt{rln}_{\mathfrak{A}}(s), \mathtt{rlnall}(s))$ to show the claim. If $\mathtt{rlnall}(s) = \mathtt{rln}_A(s) = 0$, then s is a WHNF, and $s \xrightarrow{mo,cpv,*} s'$ where s' is an MWHNF and thus $\mathtt{rln}_{mo,A}(s) = 0$. Now assume $\mathtt{rlnall}(s) > 0$. If $s \xrightarrow{no,cp} s'$, where s' is a WHNF, then $s \xrightarrow{mo,cpv,*} s''$, where s'' is a WHNF, and so $\mathtt{rln}_A(s) = \mathtt{rln}_{mo,A}(s'') = 0$. In the other cases we apply a diagram from Lemma 5.12 to a prefix of $s \xrightarrow{no,k} t_k$. For diagram (24) we have $\mathtt{rln}_{\mathfrak{A}}(t) = \mathtt{rln}_{\mathfrak{A}}(s)$, $\mathtt{rlnall}(t) < \mathtt{rlnall}(s)$, and $\mathtt{rln}_A(t) = \mathtt{rln}_A(s)$ By Theorem 3.5 (4) $\mathtt{rln}_A(r) = \mathtt{rln}_A(s)$ (also for $A = \mathfrak{A}$) and $\mathtt{rlnall}(r) < \mathtt{rlnall}(s)$. We apply the induction hypothesis to r and get $\mathtt{rln}_{mo,A}(r) = \mathtt{rln}_A(s)$ and thus $\mathtt{rln}_{mo,A}(s) = \mathtt{rln}_A(s)$.

If diagram (22), (23), or (21) is applied, then $\mathtt{rln}_{\mathfrak{A}}(t) = \mathtt{rln}_{\mathfrak{A}}(s) - 1$ and Theorem 3.5 shows that $\mathtt{rln}_A(r) = \mathtt{rln}_A(t)$. Let $\xrightarrow{no,a}$ be the given normal order reduction which is not a (no, cp). Applying the induction hypothesis to r shows $\mathtt{rln}_{mo,A}(r) = \mathtt{rln}_A(s) - 1$ (if $a \in A$) or $\mathtt{rln}_{mo,A}(r) = \mathtt{rln}_A(s)$ (if $a \notin A$). Since $s \xrightarrow{mo,*} r$ where exactly one (mo,casecx)-, (mo,seq-c)-, or (mo,β-var)-reduction is in the sequence, where $\overline{\psi}(a)$ is this reduction, we have $\mathtt{rln}_{mo,A}(s) = \mathtt{rln}_A(s)$. □

Proposition 5.14. *For all $\emptyset \neq A \subseteq \mathfrak{A}$ and closed machine expressions s: $\mathtt{rln}_A(s) = n \implies \mathtt{mln}_A(s) = n$.*

Proof. From $\mathtt{rln}_A(s) = n$ we get $\mathtt{rln}_{mo,A}(s) = n$ by Proposition 5.13. Let $s \xrightarrow{mo,k} s'$ where s' is an MWHNF. By induction on k, we show that for every reachable machine state Q_0 with $\phi(Q_0) = s$ there exists an accepting state Q_m s.t. $Q_0 \xrightarrow{*} Q_m$ and $\mathtt{mln}_A(Q_0) = n$. If $k = 0$, then $Q_0 \xrightarrow{\text{Letrec},0\vee1} Q'$ where Q' is accepting. For $k > 0$, let $s \xrightarrow{mo} s_0 \xrightarrow{mo,k-1} s'$. The following diagrams (where (UL) is (Unwind) $\vee$ (Lookup)) show the relationship between $s \xrightarrow{mo} s_0$ and the machine transition for Q_0:



The diagrams show that after applying the induction hypothesis to s_0 and Q_1 we have $\mathtt{rln}_{mo,A}(s) = \mathtt{mln}_A(Q_0)$. Finally, since $\phi(Q_0) = s$ for $Q_0 = \langle \emptyset \mid s \mid [] \rangle$, we have $\mathtt{mln}_A(s) = \mathtt{rln}_{mo,A}(s)$. □

By Propositions 5.10 and 5.14 and Lemma 5.5 we have:

Theorem 5.15. *For any closed $s \in Expr$ and $\emptyset \neq A \subseteq \mathfrak{A}$: $\mathtt{rln}_A(s) = \mathtt{mln}_A(\psi(s))$.*

Corollary 5.16. *The translation ψ seen as a translation from* LR *to the abstract machine of [10] in the variant presented in this paper is fully-abstract.*

5.3. Relating Essential and All Transition Steps

In this section, we compare our different length measures and relate counting the essential transition steps and counting all transition steps. In the following, we write (ULLU) for the union of (Unwind), (Letrec), (Lookup), (Update) and (SBS) for the union of (Subst), (Branch), (Seq).

Theorem 5.17. *For all closed machine expressions s with $s{\downarrow}$: $\mathtt{mlnall}(s) \leq 3 \cdot (\mathtt{size}(s) + 2) \cdot (\mathtt{mln}_{\mathfrak{A}}(s) + 1)$.*

Proof. Let $\mathtt{mln}_{\mathfrak{A}}(s) = n$ and $Q_0 = \langle \emptyset \mid s \mid [] \rangle \xrightarrow{m} Q_m$, where Q_m is an accepting state and the sequence contains n (SBS)-transitions. The number of (Update)-transitions is equal to the number of (Lookup)-transitions. The number of (Unwind)-transitions is equal to the number of (SBS)-transitions. It remains to count the (Letrec)- and the (Lookup)-transitions. First observe that `letrec`-expressions and -bindings which are generated by an (SBS)-transition are a copy of a subexpression which exists in s (where variables may be permuted). Since a (Letrec)-transition removes the `letrec`, there are at most $(n + 1) \cdot \mathtt{size}(s)$ (Letrec)-transitions. The same argument applies to the *first* (Lookup)-transition of a binding $x = \ldots$. The number of other (Lookup)-transitions (which are not the first for a binding $x = \ldots$) is bounded by $n + 1$, since for such a (Lookup) the binding must be $x = v$, where v is a value, which implies, that no other

(Lookup) transition can follow before another (SBS)-transition is performed. Thus, in total there are at most $(n+1)\cdot(\mathtt{size}(s)+1)$ (Lookup)-transitions.

Concluding, in the sequence there are at most $(n+1)\cdot\mathtt{size}(s)$ (Letrec)-transitions, at most $(n+1)\cdot(\mathtt{size}(s)+1)$ (Lookup)-transitions, at most $(n+1)\cdot(\mathtt{size}(s)+1)$ (Update)-transitions, and exactly n (Unwind)-transitions. By adding the n (SBS)-transitions this shows $\mathtt{mlnall}(s) \leq 3\cdot(n+1)\cdot(\mathtt{size}(s)+2)$. □

We analyze whether the measure $\mathtt{mlnlook}(\cdot)$ is appropriate as claimed in [10] and used in [5].

Proposition 5.18. *For all closed machine expressions s with $s{\downarrow}$:* $\mathtt{mlnall}(s) \leq (2\cdot\mathtt{size}(s)\cdot(\mathtt{mlnlook}(s)+1))$.

Proof. Consider a valid transition subsequence without a (Lookup)-transition. For every intermediate machine state $m_i = \langle\Gamma \mid s_i \mid S_i\rangle, i = 1,\ldots,n$ consider the expression $u_i = \phi(\langle\ \emptyset \mid s_i \mid S_i\rangle)$. Then $\mathtt{size}(u_i)$ is never increased by the intermediate steps, but strictly decreased by (Subst), (Branch), (Seq), (Update), and (Letrec). The maximal size of u_i is not greater than $\mathtt{size}(s)$ (as already argued), hence $\mathtt{mln}_{\mathfrak{A}}(s)$ + (number of (Update)s) + (number of (Letrec)s) is not greater than $\mathtt{size}(s)\cdot(\mathtt{mlnlook}(s)+1)$. Since the overall number of (Unwind)s is exactly $\mathtt{mln}_{\mathfrak{A}}(s)$, we obtain $\mathtt{mlnall}(s) \leq (2\cdot\mathtt{size}(s)\cdot(\mathtt{mlnlook}(s)+1))$. □

Theorem 5.19. *Let s be a closed machine expression. Then* $\mathtt{mlnlook}(s) \leq \mathtt{mlnall}(s) \leq (2\cdot\mathtt{size}(s)\cdot(\mathtt{mlnlook}(s)+1))$, *and* $\mathtt{mln}_{\mathfrak{A}}(s) \leq \mathtt{mlnall}(s) \leq 3\cdot(\mathtt{size}(s)+4)\cdot(\mathtt{mln}_{\mathfrak{A}}(s)+1)$.

Note that Theorem 5.19 justifies our claim that common subexpression elimination (also called β-expand) is an improvement in [10] and also in [5]. However, our proofs only show that this is the case if improvement is defined w.r.t. $\mathtt{mln}_A(.)$ in their calculus[5]. Note that also the size is not increased (up to the initial inverse (gc)) by common subexpression elimination.

As direct consequence of Proposition 3.3, Theorem 5.15, and Lemma 5.4 is:

Corollary 5.20. *Let s be an expression. Then for $s' = \psi(s)$:* $\mathtt{rln}_{\mathfrak{A}}(s') \leq (\mathtt{size}(s')+1)\cdot(\mathtt{rln}_{\{lbeta\}}(s')+1)$ *as well as* $\mathtt{mln}_{\mathfrak{A}}(s') \leq (\mathtt{size}(s')+1)\cdot(\mathtt{mln}_{\{lbeta\}}(s')+1)$. *With the size estimation in Lemma 5.4 this also shows:* $\mathtt{rln}_{\mathfrak{A}}(s) \leq (4+\mathit{maxArityOfConstructors}(s))\cdot\mathtt{size}(s)\cdot(\mathtt{rln}_{\{lbeta\}}(s)+1)$ *and* $\mathtt{mln}_{\mathfrak{A}}(s) \leq (4+\mathit{maxArityOfConstructors}(s))\cdot\mathtt{size}(s)\cdot(\mathtt{mln}_{\{lbeta\}}(s)+1)$.

The results in this section imply:

Theorem 5.21. *The following calculi allow asymptotically resource-preserving translations into each other: (i)* LR *with* $\mathtt{rln}_{\mathfrak{A}}$*; (ii) Moran-Sands calculus with* $\mathtt{mlnall}$*; (iii) Moran-Sands calculus with* $\mathtt{mln}_{\mathfrak{A}}$*; (iv) the Moran-Sands calculus with* $\mathtt{mlnlook}$*; (v) Moran-Sands calculus with* $\mathtt{mln}_{\{lbeta\}}$*;and (vi)* LR *with* $\mathtt{rln}_{\{lbeta\}}$*;*

Note that in LR switching from $\mathtt{rln}_{\mathfrak{A}}(.)$ to $\mathtt{rlnall}(.)$ is not resource-preserving, since there are LR-expressions s s.t. $\mathtt{rlnall}(s) \in O(\mathtt{size}(s)^n(\mathtt{rln}_{\mathfrak{A}}(s)+1))$ is false for all n (details are in [22]). However, this is not a counter argument against the LR-calculus, but only an argument against an implementation that really mimics the (lll)-reductions.

5.4. Comparison of Measures for Simplified Expressions

We show that after a simplification, the measures $\mathtt{mln}_{\mathfrak{A}}$, $\mathtt{mlnall}$, $\mathtt{mlnlook}$ are almost equivalent.

Definition 5.22. *An expression s is a* (lcpgc)-normal form*, if there are no (lll)-, (cpax)- nor (gc)-reductions (the (lcpgc)-reductions) possible for s. For every closed expression s, its* (lcpgc)-normal form $nf_{lcpgc}(s)$ *can be effectively computed by applying the reductions.*

[5] We also cannot express the improvement result for (cse) using $\mathtt{mlnall}$ and so-called ticks (see [10]), since ticks can only express additive work, but the measures $\mathtt{mln}_A(.)$ and $\mathtt{mlnall}$ differ by a factor depending on the size of the initial expression.

Lemma 5.23. *For every closed expression s its unique $nf_{lcpgc}(s)$ exists. It can be computed in polynomial time by exhaustively applying (lcpgc)-reductions. If s is a machine expression, then also $nf_{lcpgc}(s)$ is a machine-expression. The expression s is improved by $nf_{lcpgc}(s)$.*

A closed expression in (lcpgc)-normal form has the following syntactic structure: the letrec-subexpressions can only be the following: (i) s itself; (ii) the body of abstractions, (iii) alternatives of case-expressions, (iv), the second expression of a seq, (v) the immediate subexpressions of a constructor application; (v) bindings $x = y$ can only be of the form $x = x$.

Proof. Every reduction sequence using only (lcpgc)-reductions is terminating and confluent [24]. The other properties follow easily by checking the reduction rules. □

There is a better estimation in Theorem 5.17 for (lcpgc)-normal-forms s:

Theorem 5.24. *Let s be a closed (machine-)expression with $s{\downarrow}$ and that is in (lcpgc)-normal form. Then*

1. $\mathtt{mln}_{\mathfrak{A}}(s) \leq \mathtt{mlnall}(s) \leq 7 \cdot \mathtt{mln}_{\mathfrak{A}}(s) + 3$
2. $\mathtt{mlnlook}(s) \leq 2 \cdot \mathtt{mln}_{\mathfrak{A}}(s) + 1$
3. $\mathtt{mln}_{\mathfrak{A}}(s) \leq 2 \cdot \mathtt{size}(s) \cdot (\mathtt{mlnlook}(s) + 1)$
4. $\mathtt{mlnall}(s) \leq 2 \cdot \mathtt{size}(s) \cdot (\mathtt{mlnlook}(s) + 1)$

Proof. $\mathtt{mln}_{\mathfrak{A}}(s) \leq \mathtt{mlnall}(s)$ holds by definition of the measures. For showing $\mathtt{mlnall}(s) \leq 7 \cdot \mathtt{mln}_{\mathfrak{A}}(s) + 3$, let $s = s_0$ be in (lcpgc)-normal form. Let $Q_0\langle\emptyset \mid s_0 \mid []\rangle \xrightarrow{m} Q_m$, where Q_m is an accepting state, the sequence has n (SBS)-transitions, and let $Q_i = \langle H_i, s_i, S_i\rangle$.

Throughout the proof, we use the following invariant for each state Q_i, which can be verified by checking all transition rules and using the fact that s is in (lcpgc)-normal form:

> All expressions occurring in the heap and on control for each Q_i are in (lcpgc)-normal form, and there never appear bindings $x \mapsto s$ in the heap, where s is a variable different from x, or s is a `letrec`-expression.

We first consider the case $Q_i \xrightarrow{\text{Letrec}} Q_{i+1}$. Due to the above invariant, the subsequent transition applied to Q_{i+1} cannot be a (Letrec)-transition. The transitions that may generate a state $\langle H_i, s_i, S_i\rangle$, where s_i is a `letrec`-expression are only (Subst) and (Branch) due to the invariant. Thus the maximal number of (Letrec)-transitions in the sequence $Q_0 \xrightarrow{m} Q_m$ is at most $(n + 1)$.

The number of (Update)-transitions is at most $(2n+1)$ which can be justified as follows: We assign each (Update)-transition either to the final state, or to an (SBS)-transition: For $Q_i := (\Gamma, v, \#\mathsf{upd}(x) : S) \xrightarrow{\text{Update}} (\Gamma \cup \{x \mapsto v\}, v, S) := Q_{i+1}$, we consider what happens next in the state Q_{i+1}:

1. the stack S is empty, and a final state is reached. The (Update)-transition is assigned to the final state.
2. the stack S is non-empty and the top symbol of S is $\#\mathsf{app}(y)$, $\#\mathsf{seq}(y)$, or $\#\mathsf{case}(alts)$ (for some variable y, or alternatives *alts*, resp.). Then $Q_{i+1} \xrightarrow{SBS} Q_{i+2}$ and we assign the (Update)-transition to this (SBS)-transition.
3. the stack S is non-empty and the top symbol of S is $\#\mathsf{upd}(y)$. Let $Q_j := (\Gamma \cup \{y \mapsto t\}, y, S') \rightharpoonup (\Gamma, t, \#\mathsf{upd}(y) : S') := Q_{j+1}$ with $j < i$ be the (Lookup)-transition which added the $\#\mathsf{upd}(y)$-symbol s.t. $S = \#\mathsf{upd}(y) : S'$. Due to the invariant from above it is impossible that t is a variable or a `letrec`-expression. Moreover, t cannot be a value, since then the state Q_i cannot be of the form $(\Gamma, v, \#\mathsf{upd}(x) : \#\mathsf{upd}(y) : S)$ (since $Q_{j+1} \rightharpoonup Q_{j+2}$ would update the binding for y). Thus t can only be an application, a case-expression, or a seq-expression. In any case there must be an (SBS)-transition between Q_{j+1} and Q_i which evaluates t. We assign the (Update)-transition to this (SBS)-transition.

Considering all (SBS)-transitions, each (SBS)-transition gets at most two assignments: at most one of type (2) and also at most one of type (3), since there is at most one (Lookup)-(Update)-pair for the binding $y \mapsto t$. Thus, there are at most n (Update)-transitions of type (2), at most n (Update)-transitions of type (3) and at most 1 (Update)-transition of type (1).

Since the number of (Unwind)-transitions is exactly n (since every (SBS)-transition requires a preceding (Unwind)-transition), and since the number of (Lookup)-transitions is equal to the number of (Update)-transitions, this shows $\texttt{mlnall}(s) \leq 7n + 3$.

The second part follows from the already provided arguments. The third and fourth part follow from Theorem 5.19. □

Remark 5.25. *The factor* $\texttt{size}(s)$ *in the last two parts of the preceding theorem cannot be omitted. Let* $s_j^0 := x_j$ *and* $s_j^i := (\lambda y.s_j^{i-1})\ x_0$*, and* $t_{m,n} = \texttt{letrec}\ x_0 = \texttt{True}, \{x_i = s_{i-1}^n\}_{i=1}^m\ \texttt{in}\ x_m$*. Then* $t_{m,n}$ *is in (lcpgc)-normal form, and* $\texttt{mln}_{\mathfrak{A}}(t_{m,n}) = \texttt{mln}_{\{lbeta\}}(t_{m,n}) = m \cdot n$*,* $\texttt{mlnlook}(t_{m,n}) = m + 1$*,* $\texttt{mlnall}(t_{m,n}) = 2m \cdot (n + 1) + 3$*, and* $\texttt{size}(t_{m,n}) = m \cdot (3n + 1) + 2$*, which shows that there does not exist a polynomial poly, s.t. for all* $n \geq n_0$ *for some fixed* n_0 *and fixed* m*:* $\texttt{mlnall}(t_{m,n}) \leq poly(\texttt{mlnlook}(t_{m,n}))$ *(*$\texttt{mln}_{\mathfrak{A}}(t_{m,n}) \leq poly(\texttt{mlnlook}(t_{m,n}))$*, resp.)*

As a further remark, we show that the relation between $\texttt{mln}_{\mathfrak{A}}$ *and* $\texttt{mln}_{\{lbeta\}}$ *is not necessarily linear: For* $r_{m,n} := \texttt{letrec}\ x{=}\lambda y.(\texttt{letrec}\ z_1{=}\texttt{seq True}\ y, \{z_i{=}\texttt{seq True}\ z_{i-1}\}_{i=2}^m), u_0{=}\texttt{True}, \{u_i{=}(x\ u_{i-1})\}_{i=1}^m\ \texttt{in}\ u_n$*, The number of (lbeta)-reductions is* n *and the number of seq-reductions is* $n \cdot m$*, where* n, m *can be chosen independently. Hence, in this series of examples the inequation* $\texttt{mln}_{\mathfrak{A}}(s) \leq c \cdot \texttt{mln}_{\{lbeta\}}(s)$ *can be refuted for every constant* c*.*

5.5. Invariance of Complexity of Functions

In this section we want to analyze the relation between complexity of functions, improvement, and different measures. We will apply the analysis to machine expressions and use the mln-measures.

We will use *linear real-valued functions* $f : \mathbb{R} \to \mathbb{R}$ of the form $f\ x = a_1 \cdot x + a_2$, where $a_1, a_2 \in \mathbb{R}$ and $a_1, a_2 \geq 0$; The composition of two linear real-valued functions is again a linear real-valued function.

Definition 5.26. *Let the weak improvement relation* $\preceq_{weak}$ *be defined by* $s \preceq_{weak} t$ *iff* $s \sim_c t$ *and there is a linear real-valued function* k*, s.t. for all contexts* C*:* $\texttt{mln}_{\mathfrak{A}}(C[s]) \leq k(\texttt{mln}_{\mathfrak{A}}(C[t]))$*.*

Note that this relation is a precongruence, and that $s \preceq t$ implies $s \preceq_{weak} t$.

The claim that we show in the following is that the complexity of functions on data is not made worse under weak improvements. Roughly, the complexity of a function expression f is defined as a positive real function g, s.t. for all arguments a, $\texttt{mln}_{\mathfrak{A}}(f\ a) \leq g(\texttt{size}(a))$.

We will first analyze unary functions that operate on data, i.e. on arguments that do not contain function expressions or abstractions.

Definition 5.27. *A* data expression *in* LR *is a closed expression that consists only of constructors.*

Thus a data expression may be a constant like `True`, a pair of constants, or a finite list of data expressions.

Definition 5.28. *Let us define an upper* complexity-bound *of a closed expression* s *as a real-valued function* g*, s.t. there exists a linear real-valued function* k *and for all data expressions* d*:* $\texttt{mln}_{\mathfrak{A}}(s\ d) \leq k(g(\texttt{size}(d)))$*.*

Proposition 5.29. *If* $s \preceq_{weak} t$*, then every complexity-bound of* t *is also a complexity-bound of* s*.*

Proof. Let g be a complexity-bound of t. Then there exist linear functions k_1, k_2, such that for all data expressions d: $\texttt{mln}_{\mathfrak{A}}(s\ d) \leq k_1(\texttt{mln}_{\mathfrak{A}}(t\ d))$, and $\texttt{mln}_{\mathfrak{A}}(t\ d) \leq k_2(g(\texttt{size}(d)))$. Then for all data expressions d: $\texttt{mln}_{\mathfrak{A}}(s\ d) \leq k_1(\texttt{mln}_{\mathfrak{A}}(t\ d)) \leq k_1(k_2(g(\texttt{size}(d))))$, and $(k_1 \circ k_2)$ is also a linear function. Hence g is also a complexity-bound of s. □

Lemma 5.30. *Let* s *be a machine expression and* $\mu, \mu' \in \{\texttt{mln}_{\mathfrak{A}}(\cdot), \texttt{mlnall}(\cdot), \texttt{mlnlook}(\cdot), \texttt{mln}_{\{lbeta\}}(\cdot)\}$ *and* d *be a (machine-) data expression i.e.* $d = \phi(d')$ *for a data expression* d'*. Then there are linear real-valued functions* h_1, h_2*, such that* $\mu(s\ d) \leq h_2(\texttt{size}(s)) \cdot h_1(\mu'(s\ d))$*.*

Proof. This can be derived mainly from Section 5.3. But an additional argument is required: Applying the formula there would lead to a $h(\texttt{size}(s\ d))$ factor instead of $h(\texttt{size}(s))$. However, the data expressions do not contribute to this size component, since the contribution comes from copying abstraction. But there are no abstractions in d, hence the size of any abstraction in $(s\ d)$ is not greater than the size of s. The nontrivial cases can be derived from Theorem 5.17 and Proposition 5.18, and for $\texttt{mln}_{\{lbeta\}}$, from Proposition 3.3. □

Theorem 5.31. *Upper complexity-bounds of* LR*-functions on data are the same for all the measures* $\{\mathtt{mln}_{\mathfrak{A}}(\cdot), \mathtt{mlnall}(\cdot), \mathtt{mlnlook}(\cdot), \mathtt{mln}_{\{lbeta\}}(\cdot)\}$.

Proof. We apply Lemma 5.30 for an expression s and $\mu, \mu' \in \{\mathtt{mln}_{\mathfrak{A}}(\cdot), \mathtt{mlnall}(\cdot), \mathtt{mlnlook}(\cdot), \mathtt{mln}_{\{lbeta\}}(\cdot)\}$. There are linear real-valued functions h_1, h_2, such that for all data expressions d: $\mu(s\ d) \leq h_2(\mathtt{size}(s)) \cdot h_1(\mu'(s\ d))$. If g is a complexity upper bound of s w.r.t. μ', then there is a linear real-valued function k_1 with $\mu'(s\ d) \leq k_1(g(\mathtt{size}(d)))$. Combining this, we obtain the inequation: $\mu(s\ d) \leq h_2(\mathtt{size}(s)) \cdot h_1(k_1(g(\mathtt{size}(d)))$. Since s was chosen fixed, $h_2(\mathtt{size}(s))$ is a constant, hence there is a linear real-valued function k_2 with $\mu(s\ d) \leq k_2(\mathtt{size}(d))$. This argument is valid for all pairs μ, μ', hence the theorem holds. □

Extending the analysis to higher-order functions where the arguments d are general, i.e. may also contain abstractions, leads to weaker estimations. However, a natural condition on the expression s and argument d, implies that also in this case, the same bounds are valid. We extend Definition 5.28:

Definition 5.32. *Let s be an expression and let P be a predicate on closed expressions. An upper (higher-order)* complexity-bound *of a (closed) expression s under condition P is a real-valued function g, s.t. there is a linear real-valued function k_1 and for* all closed expressions d that satisfy P: $\mathtt{mln}_{\mathfrak{A}}(s\ d) \leq k_1(g(\mathtt{size}(d)))$.

Proposition 5.29 still holds for the higher-order complexity bounds and all P, which can be proved using the same estimations. However, Theorem 5.31 is weakened: by switching the measure, the upper higher-order complexity bounds may increase from g to $\lambda x.x \cdot g(x)$, i.e. for polynomial complexities, the degree may increase by 1 at most. For example, consider switching from $\mathtt{mln}_{\mathfrak{A}}$ to $\mathtt{mlnall}$. Let g be the higher-order complexity bound. The inequation becomes $\mathtt{mlnall}((s\ d)) \leq 3 \cdot (\mathtt{size}(s\ d) + 2) \cdot (\mathtt{mln}_{\mathfrak{A}}((s\ d)) + 1)) = 3 \cdot (3 + \mathtt{size}(s) + \mathtt{size}(d)) \cdot (\mathtt{mln}_{\mathfrak{A}}((s\ d)) + 1)) \leq 3 \cdot (3 + \mathtt{size}(s) + \mathtt{size}(d)) \cdot (k_1(g(\mathtt{size}(d)))))$. We see that the asymptotical upper bound is $a_1 + a_2 \cdot \mathtt{size}(d) \cdot g(\mathtt{size}(d))$. Thus for higher-order functions, the complexity may be increased by one order if the complexity is polynomial.

Using the estimations in Theorem 5.24, we can show that the complexity also of higher-order functions is not changed under switching between $\mathtt{mln}$ and $\mathtt{mlnall}$ under some natural conditions on the syntactic argument structure:

Lemma 5.33. *Let $\mu, \mu' \in \{\mathtt{mln}_{\mathfrak{A}}(\cdot), \mathtt{mlnall}(\cdot)\}$, s be a machine expression in lcpgc-normal form (see Definition 5.22), and d be an lcpgc-normal form. Then there is a linear real-valued function h, such that $\mu(s\ d) \leq h(\mu'(s\ d))$.*

Proof. Since $(s\ d)$ is also in lcpgc-normal form, this follows from Theorem 5.24. □

Note that for the measure $\mathtt{mlnlook}(\cdot)$ such a result cannot be obtained: due to the additional factor $\mathtt{size}(s)$ in Theorem 5.24, see Remark 5.25.

Theorem 5.34. *Let s be a closed machine expression in lcpgc-normal form, and let P be a predicate for arguments, and assume that there exists a complexity-bound according to Definition 5.32. Then the complexity-bounds of s are the same for the measures $\mathtt{mln}_{\mathfrak{A}}(\cdot)$ and $\mathtt{mlnall}(\cdot)$.*

Proof. Let $\mu, \mu' \in \{\mathtt{mln}_{\mathfrak{A}}(\cdot), \mathtt{mlnall}(\cdot)\}$. Lemma 5.33 shows that there is a linear real-valued function h s.t. $\mu(s\ d) \leq h(\mu'(s\ d))$. Let g be a complexity upper bound of s w.r.t. μ' and P. Then there is a linear real-valued function k with $\mu'(s\ d) \leq k(g(\mathtt{size}(d)))$. Combining both inequations yields $\mu(s\ d) \leq h(k(g(\mathtt{size}(d))))$. Since $h \circ k$ is a linear real-valued function, this shows that g is an upper complexity bound of s w.r.t. μ and P. Since μ, μ' where chosen freely, the theorem holds. □

6. Adding Polymorphic Typing

In this section we consider the polymorphically typed variant LRP of the calculus LR, and look for improvement in the typed setting. Considering the typed case is motivated by the fact, that there are equivalences and improvements which hold in LRP, but do not hold in LR: For the test in the definition of contextual equivalence and in the improvement relation a smaller set of contexts is taken into account,

$$
\begin{array}{rl}
\tau \in \mathit{Typ} & := a \mid (\tau_1 \to \tau_2) \mid K\ \tau_1 \ldots \tau_{ar(K)} \\
\rho \in \mathit{PTyp} & := \tau \mid \boldsymbol{\lambda} a.\rho \\
u \in \mathit{PExpr}_F & := \Lambda a_1 \ldots . \Lambda a_k . \lambda x :: \tau . s,\ k \geq 0 \\
r, s, t \in \mathit{Expr} & := u \mid x :: \rho \mid (s\ \tau) \mid (s\ t) \mid (c_{K,i} :: \tau\ s_1\ \ldots\ s_{ar(c_{K,i})}) \mid (\texttt{seq}\ s\ t) \\
& \quad \mid (\texttt{letrec}\ x_1 :: \rho_1 = s_1, \ldots, x_n :: \rho_n = s_n\ \texttt{in}\ t) \\
& \quad \mid (\texttt{case}_K\ s\ (c_{K,1} :: \tau_1\ x_{1,1} :: \tau_{1,1} \ldots x_{1,ar(c_{K,1})} :: \tau_{1,ar(c_{K,1})}\ \texttt{->}\ t_1) \quad \ldots \\
& \qquad\qquad (c_{K,|D_K|} :: \tau_{|D_K|}\ x_{|D_K|,1} :: \tau_{|D_K|,1} \ldots x_{|D_K|,ar(c_{K,|D_K|})} :: \tau_{|D_K|,ar(c_{K,i})}\ \texttt{->}\ t_{|D_K|}))
\end{array}
$$

Figure 12: Types $\tau \in \mathit{Typ}$, polymorphic types $\rho \in \mathit{PTyp}$, polymorphic abstractions $u \in \mathit{PExpr}_F$, and expressions $r, s, t \in \mathit{Expr}$ of the language LRP where $x, x_i \in \mathit{Var}$ are term variables and $a, a_i \in \mathit{TVar}$ are type variables.

$$
\frac{s :: \tau_2}{(\lambda x :: \tau_1 . s) :: \tau_1 \to \tau_2} \qquad \frac{s :: \rho}{\Lambda a . s :: \boldsymbol{\lambda} a . \rho} \qquad \frac{s :: \boldsymbol{\lambda} a . \rho}{(s\ \tau) :: \rho[\tau/a]} \qquad \frac{s :: \tau_1 \to \tau_2 \quad t :: \tau_1}{(s\ t) :: \tau_2} \qquad \frac{s :: \tau \quad t :: \tau'}{(\texttt{seq}\ s\ t) :: \tau'}
$$

$$
\frac{s :: \tau_1 \quad \mathit{pat}_i :: \tau_1 \quad t_i :: \tau_2}{(\texttt{case}_K\ s\ (\mathit{pat}_1\ \texttt{->}\ t_1) \ldots (\mathit{pat}_{|D_K|}\ \texttt{->}\ t_{|D_K|})) :: \tau_2} \qquad \frac{s_1 :: \rho_1 \quad \ldots \quad s_n :: \rho_n \quad t :: \rho}{(\texttt{letrec}\ x_1 :: \rho_1 = s_1, \ldots, x_n :: \rho_n = s_n\ \texttt{in}\ t) :: \rho}
$$

$$
\frac{\begin{array}{c} s_1 :: \tau_1, \ldots, s_{ar(c)} :: \tau_{ar(c)} \quad \tau = \tau_1 \to \ldots \to \tau_{ar(c)} \to \tau_{ar(c)+1} \\ \mathit{type}(c) = \boldsymbol{\lambda} a_1, \ldots, a_m . \tau'' \qquad \text{there are } \tau'_1, \ldots, \tau'_m \text{ with } \tau''[\tau'_1/a_1, \ldots, \tau'_m/a_m] = \tau \end{array}}{(c :: \tau\ s_1 \ldots s_{ar(c)}) :: \tau_{ar(c)+1}}
$$

Figure 13: Typing Rules for LRP

since only those contexts need to be considered which leave the expressions well-typed. Since LRP is a core language of (pure) Haskell [9] our results are applicable there. As we show, our results and techniques on improvements can straightforwardly be transferred from LR to LRP.

The extensions of LRP w.r.t. LR are type annotations at variables and constructors, and extra language components, e.g. types as arguments, including a type reduction. This will be in system-F-style and restricted to let-polymorphism [3, 13, 12, 27]. See also [19] for an analysis of typed LR of simulations as a tool for correctness. The syntax of the calculus LRP is defined in Fig. 12, where every data constructor $c \in D_K$ has a polymorphic type $\mathit{type}(c)$ of the form $\boldsymbol{\lambda} a_1, \ldots a_k . \tau_1 \to \ldots \tau_{ar(c)} \to K(a_1, \ldots, a_k)$. We assume the distinct variable convention for term and type variables. For simplicity, LRP is explicitly typed and thus we assume that all typing information is already given by the type annotations. Thus, the typing rules given in Fig. 13 only perform type checking (but not type-inference). All expressions of a polymorphic type $\boldsymbol{\lambda} a . \rho$ are of the form $x :: \rho$, $\Lambda . e$, $(e\ \tau)$, and $(\texttt{letrec}\ \mathit{Env}\ \texttt{in}\ e)$, other forms are not possible. A *polymorphic abstraction* is an expression of the form $\Lambda a_1, \ldots, a_k . \lambda x . e$, and *values* are abstraction, polymorphic abstractions, and constructor applications.

Definition 6.1. *The reduction rules of the calculus* LRP *are:*

$$
\textit{(Tbeta)} \quad ((\Lambda a . u)^{\mathsf{sub}}\ \tau) \to u[\tau/a]
$$

and all other rules from LR *(Fig. 3), extended by types and the extended syntax such that: every variable is labeled with a type, and fresh variables in rules are labeled with a type, which is derived in the rules (case-in) and (case-e) from the types of the t_i such that the types in the binding $x_i = t_i$ are equal, and in rule (cp) also polymorphic abstractions can be copied.*

The labeling algorithm is the same as for LR (see Fig. 2) where type applications are treated like usual applications. If the labeling algorithm terminates without `Fail`, then either a normal order redex is found, which is a superterm of the `sub`-marked subexpression, or the evaluation is already finished (a WHNF). Reduction contexts, weak reduction contexts, surface and top contexts are as for LR, extended by typing.

Definition 6.2 (Normal Order Reduction in LRP)**.** *Let t be an expression. Then a single normal order reduction step $\xrightarrow{\text{LRP}}$ is defined by first applying the labeling algorithm to t, and if the labeling algorithm*

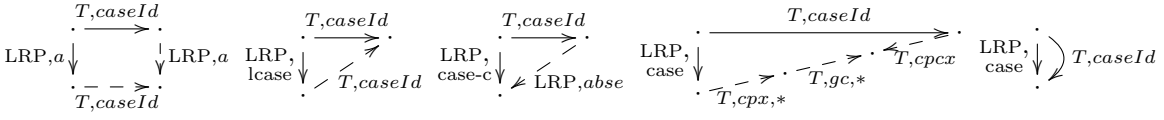


Figure 14: Diagrams for (caseId)

terminates successfully, then one of the rules in Definition 6.1 has to be applied, if possible, where the labels sub, vis *must match the labels in the expression* t.

Definition 6.3. *A* weak head normal form (WHNF) *in* LRP *is a value, or an expression of the form* (`letrec` *Env* `in` v)*, where* v *is a value, or of the form* (`letrec` $x_1 = (c\ \overrightarrow{t}), \{x_i = x_{i-1}\}_{i=2}^{m}, Env$ `in` x_m)*.*

An LRP*-expression* s converges*, denoted as* $s{\downarrow}$*, iff there exists a WHNF* t *such that* $s \xrightarrow{\text{LRP},*} t$*. Let* s, t *be two* LRP*-expressions of the same type* ρ*. Then* s *and* t *are* contextually equivalent *(denoted by* $s \sim_c t$*), iff for all contexts* $C[\cdot :: \rho]$*:* $C[s]{\downarrow} \iff C[t]{\downarrow}$.

Contextual equivalence satisfies the type substitution properties of logical relations (see e.g. [13]), i.e.: If $s \sim_c t$ with $s, t :: \rho$, then $s[\tau'/a] \sim_c t[\tau'/a]$ for $s, t :: \rho[\tau/a]$, and if $s, t :: \boldsymbol{\lambda} a.\rho$ with $s \sim_c t$, then $(s\ \tau) \sim_c (t\ \tau)$.

Definition 6.4. *The type erasure function* ϵ : LRP $\to$ LR *maps* LRP*-expressions to* LR*-expressions by removing the types, the type information and the* Λ*-construct. In particular:* $\varepsilon(s\ \tau) = \varepsilon(s)$, $\varepsilon(\Lambda a.s) = \varepsilon(s)$, $\varepsilon(x :: \rho) = x$*, and* $\varepsilon(c :: \rho) = c$.

Clearly, $\xrightarrow{\text{LRP}}$-reductions are mapped by ε to LR-normal-order reductions where exactly the $(Tbeta)$-reductions are omitted. The translation ε is not fully abstract, but adequate:

Proposition 6.5. *The translation* ε *is adequate (i.e.* $\varepsilon(e_1) \sim_c \varepsilon(e_2) \implies e_1 \sim_c e_2$*) and resource-preserving.*

The measure for estimating the time consumption of computation also in LRP is $\mathtt{rln}_A(t)$ for $\emptyset \neq A \subseteq \mathfrak{A} = \{lbeta, case, seq\}$. We do not count (TBeta)-reductions.

Definition 6.6. *Let* s, t *be two* LRP*-expressions of the same type* ρ*. We define the improvement relation* $\preceq_A$ *for* LRP*: Let* $s \preceq_A t$ *iff* $s \sim_c t$ *and for all contexts* $C[\cdot :: \rho]$*: if* $C[s], C[t]$ *are closed, then* $\mathtt{rln}_A(C[s]) \leq \mathtt{rln}_A(C[t])$*. If* $s \preceq_A t$ *and* $t \preceq_A s$*, we write* $s \approx_A t$.

The following facts are valid and can easily be verified:

1. for closed LRP-expressions s, the equation $\mathtt{rln}_A(s) = \mathtt{rln}_A(\varepsilon(s))$ holds,
2. the reduction rules and extra transformations in their typed forms can also be used in LRP. They are correct program transformations and improvements,
3. common subexpression elimination applied to well-typed expressions is an improvement in LRP.

For $\xi \in \{\leq, =, \geq\}$ and a class of contexts X we define: For s, t of type ρ the relation $s \bowtie_{\xi,X,A} t$ (in LRP) holds iff for all X-contexts $X[\cdot : \rho]$: if $X[s], X[t]$ are closed, then $\mathtt{rln}_A(X[s])\ \xi\ \mathtt{rln}_A(X[t])$. The context lemma for improvement also holds for LRP with almost the same proof.

Lemma 6.7 (Context Lemma for Improvement)**.** *Let* s, t *be* LRP*-expressions of type* ρ*. Then* $s \bowtie_{\xi,R,A} t$ *(or* $s \bowtie_{\xi,S,A} t$ *or* $s \bowtie_{\xi,T,A} t$*) implies* $s \bowtie_{\xi,C,A} t$.

We end this section by showing that the following transformation (caseId) is an improvement in LRP:

$$\text{(caseId)} \qquad (\mathtt{case}_K\ s\ (pat_1\ \mathtt{->}\ pat_1) \ldots (pat_{|D_K|}\ \mathtt{->}\ pat_{|D_K|})) \to s$$

The rule (caseId) is the heart also of other type-dependent transformations, and it is only correct under typing, i.e. in LRP, but not in LR, which can be seen by trying the case $s = \lambda x.t$.

Lemma 6.8. *Let* $s \xrightarrow{T,caseId} t$. *If* s *is a WHNF, then* t *is a WHNF. If* t *is a WHNF, then* $s \xrightarrow{\text{LRP},lll,*} \xrightarrow{\text{LRP},case,0\vee1} \xrightarrow{\text{LRP},lll,*} s'$ *where* s' *is a WHNF.*

Lemma 6.9. *If* $s\downarrow \wedge\, s \xrightarrow{T,caseId} t$, *then* $t\downarrow$ *and* $\mathtt{rln}_A(s) \geq \mathtt{rln}_A(t)$.

Proof. Let $s \xrightarrow{T,caseId} t$ and $s \xrightarrow{\text{LRP},k} s'$ where s' is a WHNF. We use induction on k. For $k = 0$, Lemma 6.8 shows the claim. For the induction step, let $s \xrightarrow{\text{LRP}} s_1$. The diagrams in Fig. 14 describe all cases how the fork $s_1 \xleftarrow{\text{LRP}} s \xrightarrow{T,caseId}$ can be closed. For diagram (25) we apply the induction hypothesis to $s_1 \xrightarrow{T,caseId} t_1$ which shows $t_1\downarrow$, $\mathtt{rln}_A(s_1) \geq \mathtt{rln}_A(t_1)$ and thus also $t\downarrow$ and $\mathtt{rln}_A(s) \geq \mathtt{rln}_A(t)$. For diagram (26) the induction hypothesis shows the claim. For diagram (27) we have $t\downarrow$, since (abse) is correct. Moreover, $t \xrightarrow{T,abse} s'$ is equivalent to $s' \xrightarrow{T,ucp\vee gc,*} t$ and Theorem 3.5 (4) and (6) show $\mathtt{rln}_A(s') = \mathtt{rln}_A(t)$. Thus also $\mathtt{rln}_A(s) \geq \mathtt{rln}_A(t)$. For diagram (28) we have $t\downarrow$, since (cpcx),(gc), and (cpx) are correct. Theorem 3.5 shows that $\mathtt{rln}_A(s) \geq \mathtt{rln}_A(s') = \mathtt{rln}_A(t)$, since (cpcx),(cpx) and (gc) do not change the measure $\mathtt{rln}_A(\cdot)$. For diagram(29) the claim obviously holds. □

Theorem 6.10. *(caseId) is an improvement.*

Proof. Lemma 6.8 and the diagrams in Fig. 14 can be used to show (by induction on the sequence for t) that if $s \xrightarrow{T,caseId} t$ and $t\downarrow$, then $s\downarrow$, since the used existentially quantified transformations are correct and diagram 26 can only be applied finitely often. Then the context lemma for $\sim_c$ (which states that convergence preservation and reflection in reduction contexts suffices to show $\sim_c$, see e.g. [18]) and Lemma 6.9 show that (caseId) is correct. Finally, the context lemma for improvement (Lemma 6.7) and Lemma 6.9 show that (caseId) is an improvement. □

7. Conclusion

We have developed a theory for improvements (w.r.t. the time behavior of programs) for the call-by-need functional core language LR. Based on the exact analysis of counting reduction steps w.r.t. program transformations in [24] in connection with a context lemma for improvement, we were able to show that several local program transformations – which for instance occur during program simplification in compilers or in reasoning tasks for larger program transformations – are improvements.

As a main result we have shown that common subexpression elimination is an improvement. This novel result proves a conjecture in [10]. Moreover, it is also practically useful, since for instance, the reasoning on improvements for worker-wrapper transformations in a call-by-need language in [5] requires the property that so-called β-expansion is an improvement, which appears in [10], but the improvement property is only conjectured. Since β-expansion is an instance of common subexpression elimination, we have also proved that β-expansion is an improvement.

Since our model for a call-by-need functional core language using a rewriting semantics is slightly different from the abstract machine model used by [10], we have deeply analyzed the connection between the formalisms. We also have clarified the relationship of the use of three different length measures, respectively, in our approach and in [10, 5]. There are differences between the measures, but they are not substantial.

A further requirement of the reasoning tasks performed in [5] is to exclude untyped cases by using a typed instead of an untyped language. Since in the typed languages more program equivalences (on typed expressions) hold, also more improvement laws hold on typed expressions. We have shown that it is rather straightforward to extend and transfer our results on improvements from the untyped language to the typed language. Additionally, we have demonstrated by a small example that our diagram-based technique can also be used in typed languages to prove that the improvement property holds for typed program transformations.

The investigation of further reasoning techniques similar to the tick-algebra of [10] is not a topic of this paper. However, recent results in [21] show that it is possible to develop such techniques which, for instance, allow to prove improvements by inductive methods for list-like structures and functions operating on them.

For future research we may investigate improvements w.r.t. other resources like space in call-by-need calculi (see [4]). We conjecture that our diagram-based proof methods can also be used in those settings. However, there are some obstacles when defining an improvement relation for space measurement. One question is which measure should be used, for instance, one can sum the size over all expressions that occur in successful reduction sequences, while it seems to be more adequate to use a measure which only counts the maximum of the size of all those expressions. A further complication is that the operational semantics of the LR-calculus as given in this paper does not care about generating and collecting garbage, i.e. `letrec`-bindings which are no longer required for computing the result. However, when measuring space this garbage should not be counted and thus it has to be removed, which requires an adapted evaluation strategy which performs garbage collection.

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